THE UNSTEADY EXPANSION OF A GAS INTO A NEAR VACUUM

Raymond McLaughlin

A Thesis Submitted for the Degree of PhD
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The unsteady expansion of a gas into a near vacuum.

by

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A Thesis submitted for the Degree of Doctor of Philosophy at the University of St. Andrews.

DECLARATION.

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been presented in application for a higher degree previously.

POSTGRADUATE CAREER.

I was admitted into the university of St. Andrews as a research student under Ordinance General No. 12 in October 1971, to carry out research work into the theory of expansions of gases into low density atmospheres under the supervision of Dr. R.E. Grundy. I was admitted under the above resolution as a candidate for the degree of Ph.D. in April 1972.

CERTIFICATE.

I certify that Raymond McLaughlin has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying application for the degree of Doctor of Philosophy.

Abstract.

This thesis is concerned with the unsteady expansion of an initially uniform, stationary gas into a low density, stationary atmosphere, studied from the viewpoint of inviscid gasdynamics.

It is found that there are two regions in the k-σ parameter space having distinct forms for the large time solution, when the atmospheric density is initially proportional to r^{-k} , r being the spatial coordinate, k being constant and σ, the geometry index, has its usual meaning. First of all a constant asymptotic shock velocity is assumed and matched expansions, for large r, are constructed. Inner expansions, valid near the shock, are matched to zeroth and first orders with the outer expansions which are valid near the contact front. Zeroth order matching, which yields the constant asymptotic shock velocity, is possible only in a restricted region of the k-σ parameter space and this situation is clarified by appealing to the similarity solutions which are extended to cover cases which have not been dealt with previously.

In the other region of the k-σ parameter space the asymptotic shock velocity is proportional to r^{ϵ} where ϵ, a positive constant, is found from the similarity solutions as a function of k, γ, σ. An attempt is made at constructing matched asymptotic expansions for large r. The inner solution can be obtained, apart from the evaluation of certain constants, to zeroth and first orders but the outer solution is inaccessible and can only be determined from the full inviscid solution. However it is shown that there exists a solution to the outer equations which matches with the inner solution up to first order. In both cases matching of the first order inner terms to the outer solution produces an eigenvalue problem, the

solution of which is not attempted here.

Finally full numerical solutions of the inviscid equations, one for each case, were produced using the method of backward drawn characteristics, devised by Hartree, and it will be seen that they compare most favourably with the asymptotic analysis.

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CONTENTS.

Chapter 1. Introduction.

Chapter 2. The problem of the near vacuum expansion.

2.1. The plane vacuum limit.

2.2. The unsteady one-dimensional expansion into a vacuum.

2.3. The unsteady one-dimensional expansion into a near vacuum.

Chapter 3. The similarity solutions in the phase plane of the zeroth order inner problem.

3.1. Introduction.

3.2. The equations and the boundary conditions.

3.3. Integral curves for $\delta = 1$.

3.4. Integral curves for $\delta > 1$.

Chapter 4. The Lagrangian formulation.

4.1. Introduction.

4.2. Definition of particle path function, outer equations and boundary equations.

4.3. The inner equations.

Chapter 5. Matched asymptotic expansions for $k < k_c$.

5.1. The outer expansions.

5.2. The breakdown of the outer solution as the shock is approached.

5.3. The inner solution and matching of the two expansions.

5.4. Determination of k_c from the similarity solution.

Chapter 6. Matched asymptotic expansions for $k > k_c$.

6.1. Investigation of possible asymptotic solutions and the evaluation of δ.

6.2. The inner expansions.

6.3. The breakdown of the inner solution.

6.4. The outer expansions and matching.

Chapter 7. Numerical solutions for finite time.

7.1. Introduction.

7.2. The characteristic equations.

7.3. Hartree's method with boundary modifications.

7.4. Stability considerations.

7.5. Numerical results and comparison with the asymptotic analysis.

Conclusions.

Appendix.

AI. Inner limit for the outer expansions, $k < k_c$.

AII. Matching conditions for the inner expansions, $k < k_c$.

1. Introduction.

Stimulated by the advent of high altitude and space flight, continuum descriptions of the motion of a gas expanding into vacuum were sought as a step towards understanding the problem of the expansion into low density atmospheres. The simplest case one can consider is that of the planar expansion of an initially uniform, stationary gas into a vacuum. Other one-dimensional motions do not afford the simplicities of planar geometry and no closed form solution of the expansion of an initially uniform cloud of gas into vacuum has been found, although we have solutions with specified non-uniform initial conditions. Keller (1956) , for example, found a class of special solutions for the one-dimensional flow of a polytropic gas and, in particular, for the expansion of a gas cloud into a vacuum. In discussing the unsteady, one-dimensional expansion of an initially uniform gas into a vacuum, Greenspan & Butler (1962) derived the important result that the gas-vacuum interface moves with constant speed for all time and for all one-dimensional motions. As a result of characteristic analysis it was shown that the gas-vacuum interface is a single limiting characteristic, unaffected by any other characteristics in the flow. Mirels & Mullen (1963) used the result of Greenspan & Butler and examined the asymptotic nature of the flow at large times. Although the analysis of Mirels & Mullen was not incorrect, Hubbard (1967) showed, in fact, that the asymptotic density profile was to a certain extent arbitrary, and went on to derive an asymptotic solution with the values of certain parameters determined by correlation with numerical solutions.

It is clear that the continuum solutions may be invalid in regions of low densities, since there the collision frequency becomes too small to support the continuum expansion. The next step, therefore, was to proceed with a non-continuum model, the starting point being Boltzmann's equation for the molecular distribution function. In some early work Brook & Oman (1965) considered the problem of a steady spherically symmetric expansion of a monatomic gas into a vacuum. However they omitted some of the convective terms in Boltzmann's equation which were later incorporated into the formulation by Hamel & Willis (1966) and Edwards & Cheng (1966). The method used by Hamel & Willis involved a procedure combining the limits of inverse Mach number and source Knudsen number approaching zero. This double limit procedure was not clearly understood but Freeman (1967) solved the problem by expanding in powers of source Knudsen number, producing the equations obtained by Hamel & Willis and Edwards & Cheng. The method of Freeman was extended by Grundy (1968) who considered the unsteady, cylindrically symmetric expansion of a finite mass of a monatomic gas and the steady, axisymmetric expansion of a monatomic gas into a vacuum.

As indicated above the eventual aim of this early work was to provide a theoretical understanding of the flow of a gas into a low density atmosphere. With this in mind it was decided , as a first step, to investigate the unsteady expansion of an inviscid gas into a near vacuum.

It is clear that a solution to this problem will not necessarily be uniformly valid throughout the flow field, but if we look upon the solution as the first term in some near continuum expansion then its relevance is plain.

In this thesis, then, we wish to examine the unsteady one-dimensional continuum expansion of an initially uniform mass of gas into a near vacuum. This examination includes the spherical, cylindrical and planar cases. We consider a uniform source gas initially at rest enclosed within $r'=L$ and surrounded by a stationary atmosphere of lower density and sound speed. In what follows we refer to the gas cloud as the source gas and the surrounding gas as the atmosphere. At time $t'=0$ the expansion is allowed to take place and the subsequent flow is studied from the viewpoint of inviscid gasdynamics.

The general picture in the r'-t' plane is well known, a contact front separates the two gases and drives the primary shock ahead of it while, in general, a secondary shock is formed behind the contact front.

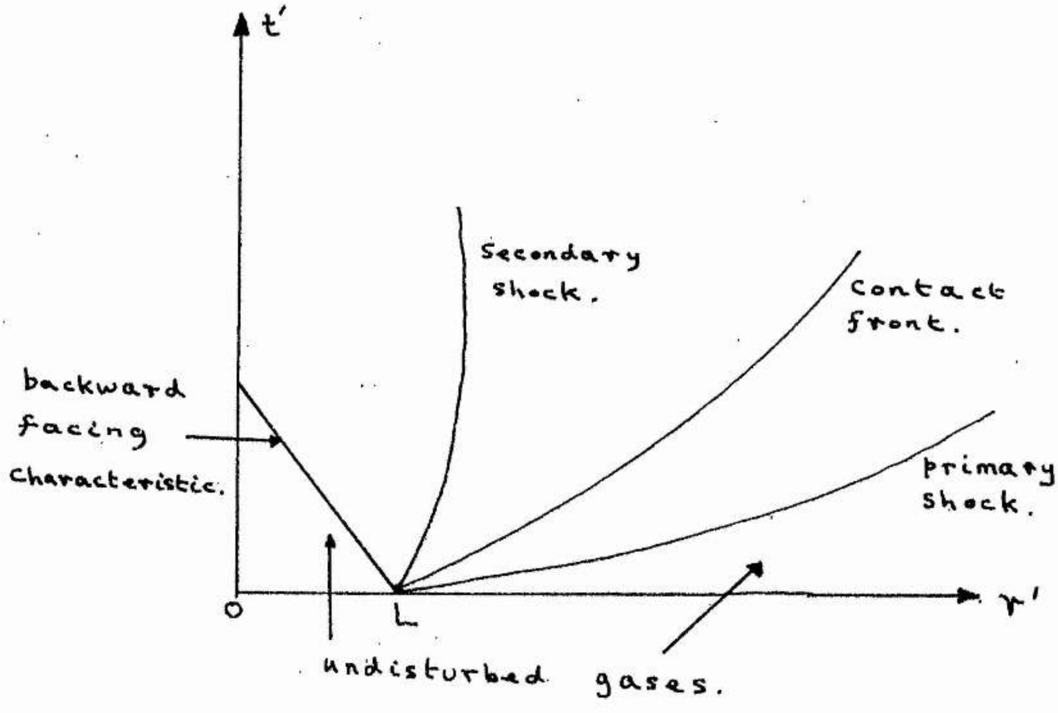


FIG. 1.

We are interested in the motion of the contact front-primary shock system and in particular the way in which the flow degenerates into the perfect vacuum expansion as the atmospheric density and sound speed go to zero.

To introduce some of the ideas we employ in the general one-dimensional case, we look at the plane shock tube problem where there is a full analytic solution available. It is shown that, as the density and sound speed of the atmosphere approach zero, the contact front between the source gas and the atmosphere moves, to a first approximation, with a velocity equal to that of the gas-vacuum interface in the corresponding expansion into a vacuum. In the conclusion of their 1962 paper, Greenspan & Butler suggest that the gas-vacuum interface could conceivably serve to locate the approximate position of a strong shock if the expansion took place into a uniform quiescent atmosphere of density and sound speed much lower than those of the source gas. Grundy (1972) took up this point, intending to clarify the rôle of the perfect-vacuum expansion in this context. Using the gas-vacuum interface as a first approximation to the contact front, and taking the limit as atmospheric density and sound speed go to zero, he constructed matched expansions for the large time solution of the Lagrangian equations and showed that the contact front and the strong shock driven by it have distinctly different motions and, contrary to the conjecture of Greenspan & Butler, do not coalesce.

In this thesis we extend the work of Grundy (1972) to cover the case of an atmospheric density initially proportional to $(r')^{-k}$, k being a positive constant. The motivation for this generalisation is its intrinsic mathematical interest and its relevance to the study of blast waves propagating into the ambient solar wind or any non-uniform low density atmosphere. The problem also has certain similarities with the asymptotic theory of hypersonic flow past blunt bodies, an interesting additional point which will be discussed later. Turning to the solar wind problem, Parker (1965) used the progressing waves or similarity solutions of Courant & Friedrichs with spherical symmetry to obtain a picture of what happens to the ambient solar wind when there is a sudden expansion of the solar corona. Parker principally considers the case k=2, since for a steady spherically symmetric expansion, under certain assumptions, this is the asymptotic density decay. The model is not strictly correct in that the velocity of the ambient solar wind is ignored in the formulation. Recently, however, Grundy (1974) has included this effect in a progressing wave formulation and produced a new shock locus.

Using methods similar to those of Grundy (1972) we study the large time behaviour. We look at the large time solution because this is what one usually is interested in and observes in a physical situation. The Lagrangian equations are chosen as a starting point because of the relative ease with which the matching procedure can be performed. For the zeroth order inner problem we eventually obtain two non-linear ordinary differential equations which, when solved with the appropriate boundary and matching conditions give the constant asymptotic shock velocity. These two equations are the Lagrangian formulation of the equations obtained by Sedov (1959) and Courant & Friedrichs (1948) in their similarity analysis.

For an asymptotically constant shock velocity there is an upper limit, k_c, to k for a successful integration of these differential equations. To clarify this point we turned to the similarity solutions and the phase plane of the zeroth order inner problem. Although Sedov gives an adequate account of these similarity solutions he does not cover the whole $k-\sigma$ parameter space and so we have to significantly extend his work in order to solve our problem for all $k>0$, $\sigma \geq 0$. We then exploit these solutions further and in the process calculate k_c as a function of γ, the ratio of the specific heats of the atmosphere, and σ, the geometry index.

In the solution of the problem for $k>k_c$, the shock path takes the asymptotic form $r'=\xi'(t')$, where ξ' and δ are constant with $\delta>1$. The index is calculated using the similarity solution to the zeroth order inner problem. In his treatment of the solutions, Sedov concentrates on $\delta<1$, no mention being made of the case $\delta>1$. We extend the work of Sedov to this case and apply the solutions to our problem. It is then apparent that δ is determined uniquely as a function of k, γ, σ; its value is calculated numerically. An attempt is made at constructing matched asymptotic expansions for $k>k_c$ using the Lagrangian formulation. The inner solution is well defined except for the evaluation of certain constants but the outer solution presents a different state of affairs. It is found that the truncation of the set of equations, which is so vital for success, is not possible. This indeterminacy has also been found by Grundy (1969) who sought solutions for the time dependent expansions of monatomic gases with sperical symmetry. Hubbard, as noted, also showed this in another description. Even so it is shown that there is a solution that will match with the inner solution, but it is clear that the outer solution is determined by the full inviscid solution and therefore by the initial conditions.

In these asymptotic solutions, both for $k<k_c$ and $k>k_c$, matching of terms in the first order inner solution with terms in the outer expansions yields an eigenvalue problem. The solution of this is not attempted but an outline of a possible method of approach, suggested by Stewartson & Thompson (1970) for the hypersonic blunt body problem, is given in chapter 5.

Finally numerical solutions for finite time were produced, using a second order finite difference approximation to the Lagrangian equations in characteristic form. The method of Specified Time Intervals or Backward Drawn Characteristics, devised by Hartree (1952), is employed with slight modifications to the procedure near to sloping boundaries. The details of this are given in chapter 7 where also the results are given. The purpose of the sample numerical integrations of the full equations is to compare them with the asymptotic theory. The computer time involved in these integrations is immense and fully justifies the asymptotic theory which requires only a fraction of the computer time to work out the shock velocity or the index δ, for example.

2. The problem of the near vacuum expansion.

2.1. The plane vacuum limit.

In this section we discuss the expansion of an inviscid gas into a near vacuum with plane geometry. This is essentially the plane shock tube problem and we show that, in the appropriate limit, the behaviour exhibited is that of the vacuum expansion. The solution to the classical shock tube problem is well documented as in, for example, Courant & Friedrichs. Here it will serve to indicate certain features that we exploit for general one-dimensional geometry. An outline of the solution is now given prior to taking the vacuum limit.

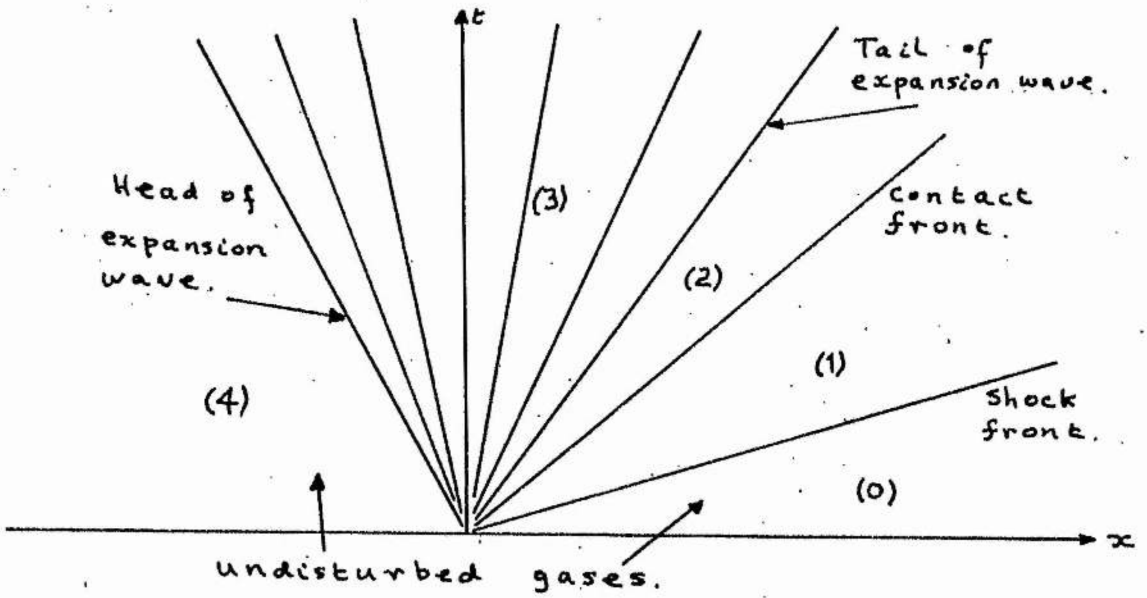
We consider a straight tube of constant cross section initially occupied for $x<0$ by a uniform stationary gas of sound speed a_4, density ρ_4, pressure p_4 and ratio of its specific heats γ_4; while the region $x>0$ contains a uniform stationary gas of sound speed a_0, density ρ_0, pressure p_0 and ratio of its specific heats γ_0, where

$$a_i^{\,2} = \frac{\gamma_i p_i}{\rho_i} \quad , \text{ for } i=0,4,$$

and $\left(\frac{a_0}{a_4}\right) << 1, \ \left(\frac{\rho_0}{\rho_4}\right) << 1.$

The expansion is allowed to take place at time $t=0$ and a schematic picture of the resulting flow is shown in FIG. 2.

Any suffices refer to regions in the x-t plane. Region (0) is the undisturbed lower density gas. Region (1) is the uniform gas between the shock front and the contact front separating the two gases. Region (2) is a uniform region between the contact front and the simple wave, region (3).



Region (4) is the undisturbed higher density gas.

The Rankine-Hugoniot relations across the shock front give

$$u_1 = \frac{2V}{(\gamma_0+1)}\left\{1-\frac{a_0^2}{V^2}\right\} \quad , \qquad (2.1.1)$$

$$\rho_1 = \rho_0\left\{\frac{(\gamma_0+1)}{(\gamma_0-1)+\frac{2a_0^2}{V^2}}\right\} \quad , \qquad (2.1.2)$$

$$p_1 = \frac{2\rho_0 V^2}{(\gamma_0+1)}\left\{1-\frac{(\gamma_0-1)a_0^2}{2\gamma_0 V^2}\right\} \quad , \qquad (2.1.3)$$

where V is the shock velocity.

In the simple wave we have

$$\left.\begin{aligned} \frac{u_3}{a_4} &= \frac{2}{(\gamma_4+1)}\left\{1+\frac{\eta}{a_4}\right\} \quad , \\ \frac{p_3}{p_4} &= \left\{\frac{2}{(\gamma_4+1)} - \frac{(\gamma_4-1)}{(\gamma_4+1)}\frac{\eta}{a_4}\right\}^{\frac{2\gamma_4}{(\gamma_4-1)}} \quad , \end{aligned}\right\} \qquad (2.1.4)$$

where $\eta = \frac{x}{t}$.

We let the equation for C_- , the tail of the expansion wave, be $\eta = \eta^*$ and employ the continuity of gas velocity and pressure between regions (3), (2) and across the contact front to arrive at two separate equations for η^*. These are

$$\frac{\eta^*}{a_4} = \frac{(\gamma_4+1)\left(\frac{a_0}{a_4}\right)\left\{\frac{p_1}{p_0} - 1\right\}}{\sqrt{2\gamma_0}\left\{(\gamma_0+1)\left(\frac{p_1}{p_0}\right) + (\gamma_0-1)\right\}^{\frac{1}{2}}} \quad , \qquad (2.1.5)$$

and

$$\frac{\eta^*}{a_4} = \frac{2}{(\gamma_4-1)} - \frac{(\gamma_4+1)}{(\gamma_4-1)}\left\{\frac{p_1}{p_0}\frac{\rho_0}{\rho_4}\frac{\gamma_4}{\gamma_0}\left(\frac{a_0}{a_4}\right)^2\right\}^{\frac{(\gamma_4-1)}{2\gamma_4}} \qquad (2.1.6)$$

The fact that $V \to \frac{(\gamma_0+1)}{(\gamma_4-1)} a_4$ as $\left(\frac{a_0}{a_4}\right)$ and $\left(\frac{\rho_0}{\rho_4}\right) \to 0$ can of course be verified from the full solution directly, but here it is easier to make the assumption. Thus we seek a perturbation of the vacuum solution by letting

$$V = \frac{(\gamma_0+1)}{(\gamma_4-1)} a_4 \left\{1 + \epsilon + \ldots\ldots\right\} \quad ,$$

where ϵ, a small quantity, will be found in terms of $\frac{a_0}{a_4}$ and $\frac{\rho_0}{\rho_4}$ later.

We have, now, from (2.1.1) and (2.1.3),

$$u_1 = \frac{2a_4}{(\gamma_4-1)}\left\{1 + \epsilon + \delta_1 + \ldots\ldots\right\} \quad , \qquad (2.1.7)$$

and

$$\frac{p_1}{p_0} = \frac{2\gamma_0(\gamma_0+1)}{(\gamma_4-1)^2}\left(\frac{a_4}{a_0}\right)^2\left\{1 + 2\epsilon + \frac{(\gamma_0-1)\delta_1}{2\gamma_0} + \ldots\right\} \quad ,$$

where $\delta_1 = -\dfrac{(\gamma_4-1)^2}{(\gamma_0+1)^2}\left(\dfrac{a_0}{a_4}\right)^2$ is small.

Using these expressions, (2.1.5) and (2.1.6) become

$$\frac{\eta^*}{a_4} = \frac{2}{(\gamma_4-1)} + \frac{(\gamma_4+1)}{(\gamma_4-1)}\left\{\epsilon + \delta_1 + \ldots\ldots\right\} , \qquad (2.1.8)$$

and

$$\frac{\eta^*}{a_4} = \frac{2}{(\gamma_4-1)} - \frac{(\gamma_4+1)}{(\gamma_4-1)}\left\{\frac{2\gamma_4(\gamma_0+1)}{(\gamma_4-1)^2}\,\frac{\rho_0}{\rho_4}\right\}^{\frac{(\gamma_4-1)}{2\gamma_4}} + \ldots , \qquad (2.1.9)$$

Also we find that

$$\lim_{\eta \to \eta^*_-}\left(\frac{a_3}{a_4}\right) = \left\{\frac{2\gamma_4(\gamma_0+1)}{(\gamma_4-1)^2}\,\frac{\rho_0}{\rho_4}\right\}^{\frac{(\gamma_4-1)}{2\gamma_4}} + \ldots \quad ,$$

where η approaches η^* from below, and a_3 is the sound speed in the simple wave region.

Comparing (2.1.8) and (2.1.9) gives

$$\epsilon = -\frac{(\gamma_4-1)^2}{(\gamma_0+1)^2}\left(\frac{a_0}{a_4}\right)^2 - \left\{\frac{2\gamma_4(\gamma_0+1)}{(\gamma_4-1)^2}\,\frac{\rho_0}{\rho_4}\right\}^{\frac{(\gamma_4-1)}{2\gamma_4}} , \qquad (2.1.10)$$

and evidently $\epsilon \to 0$ as $\left(\dfrac{a_0}{a_4}\right) \to 0$, $\left(\dfrac{\rho_0}{\rho_4}\right) \to 0$.

Thus from (2.1.7) and (2.1.8) we conclude that as both $\frac{a_0}{a_4}$ and $\frac{\rho_0}{\rho_4}$ go to zero the equation of the contact front is, to a first approximation,

$$\left.\begin{aligned} \frac{x}{t} &= \frac{2a_4}{(\gamma_4-1)} \quad , \\ \text{on which} \qquad u &= \frac{2a_4}{(\gamma_4-1)} + \ldots\ldots \end{aligned}\right\} \qquad (2.1.11)$$

together with zero sound speed to the immediate left of the contact front, with error indicated by (2.1.10).

To a first approximation, therefore, the contact front can be replaced by the gas-vacuum interface.

If there is now a rigid wall at $x = -L$, then the expansion wave front will, at time $t = \frac{L}{a_4}$, reflect from this wall. This reflected wave will not affect the previous solution for

$$x > x_r = \frac{2a_4 t}{(\gamma_4-1)} - \frac{(\gamma_4+1)}{(\gamma_4-1)} L \left(\frac{a_4 t}{L}\right)^{\frac{(3-\gamma_4)}{(\gamma_4+1)}}$$

and will not influence the motion of the contact front in the vacuum limit since

$$\frac{2a_4 t}{(\gamma_4-1)} > x_r \quad \text{for all } t.$$

This is important since in non-planar one-dimensional expansions there will be reflections of the wave from the centre of symmetry but we assume that they will not affect the motion of the contact front in the vacuum limit.

2.2. The unsteady one-dimensional expansion into a vacuum.

The result of Greenspan & Butler is fundamental to the work presented here. That the gas-vacuum interface of the expansion into vacuum, and hence the contact front in the vacuum limit, moves with constant velocity for all time is a crucial result, since it simplifies the boundary condition there. Because of the importance of this result we briefly discuss it below.

Consider an inviscid gas initially enclosed within radius $r = L$, with constant sound speed a_4, constant density ρ_4. The space $r > L$ is initially evacuated and, at time $t = 0$, the unsteady expansion is allowed to take place.

The one-dimensional equations of gasdynamics in characteristic form are

$$\frac{\partial u}{\partial \xi} + \frac{2}{(\gamma_4 - 1)} \frac{\partial a}{\partial \xi} + \frac{\sigma a u}{r} \frac{\partial t}{\partial \xi} = 0; \quad \frac{\partial r}{\partial \xi} = (u+a) \frac{\partial t}{\partial \xi}, \qquad (2.2.1)$$

$$\frac{\partial u}{\partial \eta} - \frac{2}{(\gamma_4 - 1)} \frac{\partial a}{\partial \eta} - \frac{\sigma a u}{r} \frac{\partial t}{\partial \eta} = 0; \quad \frac{\partial r}{\partial \eta} = (u-a) \frac{\partial t}{\partial \eta}, \qquad (2.2.2)$$

where γ_4 is the ratio of specific heats of the gas, assumed constant, σ, the geometry index, is 0,1,2 for plane, cylindrical or spherical symmetry respectively and ξ, η are the characteristic coordinates.

Subsequent to release, the only particles in motion lie between the gas-vacuum interface and the rarefaction front propagating into the stationary gas. The interface is the zero sound speed surface $a = 0$, along which pressure and density are both zero; the velocity of this front is unknown at this stage. The accoustic front, however, propagates with constant velocity,

a_4 , into the quiescent gas.

Since $a = 0$ on the interface we may argue that it is either a characteristic or an envelope of characteristics. Consider the latter case and let $\eta(r,t) =$ const. be a C_+ characteristic which emanates from the stationary gas and touches the interface. Upon integrating equation (2.2.1) along the entire length of this characteristic from ξ_4 to ξ_f , the front, we find that

$$u_f = \frac{2a_4}{(\gamma_4-1)} - \int_{\xi_4}^{\xi_f} \frac{\sigma au}{r} \frac{\partial t}{\partial \xi} \, d\xi \ , \qquad (2.2.3)$$

using the fact that $a = 0$, $u = u_f$ at $\xi = \xi_f$, the front, and $a = a_4$, $u = 0$ at $\xi = \xi_4$, in the undisturbed gas.

Since the front initially moves with the locally plane velocity of $\frac{2a_4}{(\gamma_4-1)}$, and, for small t at least, the term $\frac{\sigma au}{r}$ is non-negative, and $\frac{\partial t}{\partial \xi} d\xi = dt > 0$, (2.2.3) implies that the front must at some time decelerate. That this is impossible is a direct consequence of the Lagrangian equation of motion of the frontal particle,

$$\frac{\partial u_f}{\partial t} = - \frac{2}{(\gamma_4-1)} \left(a \frac{\partial a}{\partial r}\right)_f \quad . \qquad (2.2.4)$$

Now the sound speed is a non-negative quantity which increases away from the front, $a = 0$, i.e.

$$\left(a \frac{\partial a}{\partial r}\right)_f \leqslant 0 \ . \qquad (2.2.5)$$

The acceleration of the frontal particle is then non-negative, showing that it cannot decelerate at any time. Therefore the front must be a characteristic curve, not an envelope, and the velocity of the diverging interface is a constant

$$u_f = \frac{2a_4}{(\gamma_4 - 1)} \quad . \qquad (2.2.6)$$

2.3. The unsteady one-dimensional expansion into a near vacuum.

We consider a uniform source gas initially ay rest within $r' = L$ with sound speed a_4 , density ρ_4 and the ratio of its specific heats γ_4 . This gas cloud is surrounded by a stationary atmosphere of density $\rho_0' = \rho^* \left(\frac{r'}{L}\right)^{-k}$; sound speed $a_0' = a^* f\left(\frac{r'}{L}\right)$ and the ratio of its specific heats γ_0 , where ρ^*, k, a^* are positive constants and $f\left(\frac{r'}{L}\right)$ is a positive function restrictions on which will be placed later.

At time $t' = 0$ the expansion is allowed to take place and the ensuing motion of the contact front-primary shock system is studied from the viewpoint of inviscid gasdynamics in the limits $\frac{\rho^*}{\rho_4} \to 0$, $\frac{a^*}{a_4} \to 0$.

The Eulerian equations describing the unsteady one-dimensional flow of an inviscid adiabatic gas are

$$\left.\begin{aligned} &\frac{\partial}{\partial t'}(\rho' r'^{\sigma}) + \frac{\partial}{\partial r'}(\rho' u' r'^{\sigma}) = 0 \ , \\ &\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial r'} + \frac{1}{\rho'}\frac{\partial p'}{\partial r'} = 0 \ , \\ &\left\{\frac{\partial}{\partial t'} + u' \frac{\partial}{\partial r'}\right\}(p'/\rho'^{\gamma_0}) = 0 \ , \end{aligned}\right\} \qquad (2.3.1)$$

where p' , ρ' , u' , r' , t' are the dimensional pressure, density, velocity, spatial coordinate and time respectively. The geometry index, σ, has its usual meaning.

From the result of Greenspan & Butler and the plane result, expression (2.1.11), of section 2.1. we infer that the boundary condition at the contact front is

$$u' = \frac{2a_4}{(\gamma_4-1)}\left\{1 + O\left(\frac{a^*}{a_4}\right)^{\alpha_1} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_1}\right\}$$

$$\text{on} \quad r' = L + \frac{2a_4 t}{(\gamma_4-1)}\left\{1 + O\left(\frac{a^*}{a_4}\right)^{\alpha_1} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_1}\right\}, \qquad (2.3.2)$$

where α_1 and β_1 are positive constonts.

We now consider boundary conditions at the shock. With conditions in front of the shock denoted by subscript '0' and those behind by subscript '1' , the Rankine-Hugoniot relations are

$$u_1' = \frac{2V_1'}{(\gamma_0+1)}\left\{1 - \left(\frac{a_0'}{V_1'}\right)^2\right\},$$

$$\rho_1' = \rho_0'\left\{\frac{(\gamma_0+1)}{(\gamma_0-1) + 2\dfrac{a_0'^2}{V_1'^2}}\right\}, \qquad (2.3.3)$$

$$p_1' = \frac{2\rho_0' V_1'^2}{(\gamma_0+1)}\left\{1 - \frac{(\gamma_0-1)a_1'^2}{2\gamma_0 V_1'^2}\right\},$$

$$\text{on} \quad \frac{dr'}{dt'} = V_1' ,$$

where V_1' is the dimensional shock velocity.

Now for small $\frac{a^*}{a_4}$, $\frac{\rho^*}{\rho_4}$ we let

$$u' = \frac{2a_4}{(\gamma_4-1)}\left\{u + O\left(\frac{a^*}{a_4}\right)^{\alpha_2} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_2}\right\},$$

$$\left.\begin{aligned} p' &= \left(\frac{2a_4}{\gamma_4-1}\right)^2 \rho^*\left\{p + O\left(\frac{a^*}{a_4}\right)^{\alpha_3} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_3}\right\} , \\ \rho' &= \rho^*\left\{\rho + O\left(\frac{a^*}{a_4}\right)^{\alpha_4} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_4}\right\} , \\ V_1' &= \frac{2a_4}{(\gamma_4-1)}\left\{V_1 + O\left(\frac{a^*}{a_4}\right)^{\alpha_5} + O\left(\frac{\rho^*}{\rho_4}\right)^{\beta_5}\right\} , \end{aligned}\right\} \qquad (2.3.4)$$

where the α_i , β_i are positive constants.

Also we let $r' = rL$, $t' = \dfrac{Lt(\gamma_4-1)}{2a_4}$. (2.3.5)

Inserting (2.3.4), (2.3.5) in (2.3.1), (2.3.2), (2.3.3) gives, neglecting small quantities and replacing γ_0 by γ ,

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho r^\sigma) + \frac{\partial}{\partial r}(\rho u r^\sigma) = 0 , \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 , \qquad (2.3.6) \\ &\left\{\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right\}(p/\rho^\gamma) = 0 , \end{aligned}$$

together with $u = 1$ on $r = 1+t$, and (2.3.7)

$$\left.\begin{aligned} u_1 &= \frac{2V_1}{(\gamma+1)} , \\ p_1 &= \frac{2V_1^2 r^{-k}}{(\gamma+1)} , \\ \rho_1 &= \frac{(\gamma+1)}{(\gamma-1)} r^{-k} , \\ \text{on} \quad \frac{dr}{dt} &= V_1 . \end{aligned}\right\} \qquad (2.3.8)$$

The perturbations to (2.3.8) will be of the order of

$$\frac{a_0'^2}{V_1'^2} = \left(\frac{a^*}{a_4}\right)^2 \frac{f^2(r)}{V_1^2(r)} ,$$

and for this to vanish uniformly in r as $\left(\frac{a^*}{a_4}\right)^2 \to 0$ we must have that $\frac{f(r)}{V_1(r)}$ is bounded for all r. This then is the restriction which must be placed upon f(r) and to illustrate this we note the example of a polytropic gas

$$a_0'^2 = (\gamma p_0' / \rho_0') \sim \rho_0'^{(\gamma-1)} \sim r^{-k(\gamma-1)} \sim f(r) .$$

The problem is to solve (2.3.6) and find V_1 subject to (2.3.7) and (2.3.8).

3. The similarity solutions in the phase plane of the inner problem.

3.1. Introduction.

In later chapters we discuss certain features in our problem by appealing to these similarity solutions. Although Sedov gives an adequate account of the solutions in a restricted region of the $k-\sigma$ parameter space, not enough information is available for the problem under discussion. Therefore we have to significantly extend Sedov's work to deal with the cases that we have here.

We use a notation similar to that of Sedov and reproduce the relevant ordinary differential equations. We examine these equations for two cases basically, $\delta = 1$ for all k, $\delta > 1$ for $k > (\sigma+1)$, where the similarity variable is defined as $\lambda = rt^{-\delta}$. In the first of these cases we extend Sedov's work to cover the whole $k-\sigma$ parameter space. With respect to the second case, Sedov concentrates on $\delta < 1$, while we investigate the solutions for $\delta > 1$, $k > (\sigma+1)$.

3.2. The equations and boundary conditions.

Following Sedov we let

$$\left.\begin{aligned} u &= \delta \frac{r}{t} V(\lambda) , \\ p &= \delta^2 (r / t)^2 r^{-k} P(\lambda) , \\ \rho &= r^{-k} R(\lambda) , \\ a^2 &= \delta^2 (r / t)^2 Z(\lambda) , \\ \text{where } a^2 &= \gamma p / \rho , \\ Z(\lambda) &= \gamma P(\lambda) / R(\lambda) , \end{aligned}\right\} \qquad (3.2.1)$$

$\lambda = r\,t^{-\delta}$ is the similarity variable (3.2.2)

and $\delta \geqslant 1$ is a constant.

Substituting these in (2.3.6) eventually gives a single ordinary differential equation in Z and V and three others for λ, P, R in Z and V. These equations are

$$\frac{dZ}{dV} = \frac{Z\left\{\left[2(V-\frac{1}{\delta})+(\gamma-1)(\sigma+1)V\right](1-V)^2+(\gamma-1)V(V-\frac{1}{\delta})(1-V)-Z\left[2(V-\frac{1}{\delta})+\frac{K}{\delta}(\gamma-1)\right]\right\}}{(1-V)\left\{V(V-\frac{1}{\delta})(1-V) + Z\left[(\sigma+1)V - \frac{K}{\delta}\right]\right\}}, \tag{3.2.3}$$

$$\frac{d(\ln\lambda)}{dV} = \frac{(1-V)^2 - Z}{V(V-\delta^{-1})(1-V) + Z\left[(\sigma+1)V-K\delta^{-1}\right]}, \tag{3.2.4}$$

$$\frac{d(\ln P)}{dV} = \frac{\gamma V(1-V)(\sigma+1-K\delta^{-1})-2(1-V)^2\delta^{-1}+\gamma V(V-\delta^{-1})-Z(\gamma K-2)\delta^{-1}}{V(V-\delta^{-1})(1-V) + Z\left[(\sigma+1)V-K\delta^{-1}\right]}, \tag{3.2.5}$$

$$(1-V)\,\frac{d(\ln R)}{dV} = \frac{(\sigma+1-k)V\left\{(1-V)^2 - Z\right\}}{V(V-\delta^{-1}) + Z\left[(\sigma+1)V-K\delta^{-1}\right]} + 1, \tag{3.2.6}$$

where $$K = \frac{2 + \delta(k-2)}{\gamma}. \tag{3.2.7}$$

Now we investigate the positions of the boundaries, the shock and the contact front, in the phase plane of Z and V. Let the value of λ on the shock be ξ, that is, on the shock, as $t \to \infty$,

$$\left.\begin{aligned} r &= \xi t^{\delta}, \\ \text{and}\quad V_1 &= \delta\,\frac{r}{t}, \end{aligned}\right\} \tag{3.2.8}$$

V_1 being the non-dimensional shock velocity. Thus the strong shock relations, (2.3.8), become

$$\left.\begin{aligned} V &= \frac{2}{\gamma+1} \ , \\ P &= \frac{2}{\gamma+1} \ , \\ R &= \frac{(\gamma+1)}{(\gamma-1)} \ , \\ &\text{and also} \\ Z &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \ , \quad \text{on } \lambda = \xi \ . \end{aligned}\right\} \qquad (2.3.9)$$

In order to locate the coordinates of the contact front we use the fact that, as $t \to \infty$, this front follows the line

$$\frac{dr}{dt} = \frac{r}{t} = \lambda t^{\delta-1} \ , \qquad (3.2.10)$$

along which $u = \frac{r}{t}$. (3.2.11)

The information derived from (3.2.10), (3.2.11) is different for the two classes of problem chosen and so we treat them separately.

Let $\delta = 1$, then $\lambda = rt^{-1}$ and then (3.2.10), (3.2.11) together with the first of equations (3.2.1) give

$$V = 1 \ , \ \lambda = 1 \qquad (3.2.12)$$

as the contact front.

If we now look at (3.2.10) for $\delta > 1$, we see that, since $\frac{dr}{dt}$ is always finite on the contact front, then $\lambda \to 0$ as $t \to \infty$. Thus the contact front corresponds to $\lambda = 0$. Equation (3.2.11) together with the first of equations (3.2.1) gives

$V = \delta^{-1}$ on the contact front. Indeed we shall see later that the only point in the Z-V plane which can possibly be associated with the contact front is the singular point located at $Z = 0$, $V = \delta^{-1}$.

In order to facilitate the solution of equation (3.2.3) we first locate its singularities and then determine the local behaviour of the integral curves near them. We also make use of the loci of zero and infinite values of $\frac{dZ}{dV}$.

Although we are integrating the same equation for $\delta = 1$ or $\delta > 1$, there are certain features in one case not found in the other and vice-versa. Since it saves neither time nor effort to generalise from this point on, we shall discuss each of the two cases separately.

3.3. Integral curves for $\delta = 1$.

If we put $\delta = 1$ in equations (3.2.3), (3.2.4) we have

$$\frac{dZ}{dV} = \frac{Z\, S(V,Z)}{(1-V)\, Q(V,Z)} , \tag{3.3.1}$$

$$\frac{d(\ln\lambda)}{dV} = \frac{D(V,Z)}{Q(V,Z)} , \tag{3.3.2}$$

$$\left.\begin{aligned} \text{where}\quad D(V,Z) &= Z - (1-V)^2 , \\ Q(V,Z) &= V(1-V)^2 + Z\left[k/\gamma - V(\sigma+1)\right] , \qquad (3.3.3) \\ S(V,Z) &= 2(1-V)^3 - \sigma(\gamma-1)V(1-V)^2 - Z\left[2(1-V) - k(\gamma-1)/\gamma\right] . \end{aligned}\right\}$$

Also if we put $\hat{Z} = Z^{-1}$, we can investigate the behaviour of the integral curves at large Z using

$$\frac{d\hat{Z}}{dV} = \frac{-\hat{Z}\,\hat{S}(V,\hat{Z})}{(1-V)\,\hat{Q}(V,\hat{Z})} \quad ,$$

$$\frac{d(\ln\lambda)}{dV} = \frac{\hat{D}(V,\hat{Z})}{\hat{Q}(V,\hat{Z})} \quad , \qquad (3.3.4)$$

where

$$\hat{D}(V,\hat{Z}) = \hat{Z}\,D(V,Z) \quad ,$$
$$\hat{Q}(V,\hat{Z}) = \hat{Z}\,Q(V,Z) \quad , \qquad (3.3.5)$$
$$\hat{S}(V,\hat{Z}) = \hat{Z}\,S(V,Z) \quad .$$

The singularities of equations (3.3.1) and (3.3.2) are found in the usual way and a full list appears in TABLE 1.

We are only interested in solutions for $k > 0$ and therefore can ignore singularity H. Also, since $\frac{dZ}{dV}$ is infinite on $V = 1$, $Z > 0$, the solution curve lies in the section of the plane $Z \geqslant 0$, $0 \leqslant V \leqslant 1$. Thus we can disregard singularity I for all k, singularity B if $Z_B < 0$ and singularity A if $k > \gamma\sigma$, that is $V_A > 1$.

Now we look at the behaviour of the integral curves of (3.3.1), (3.3.2) in the neighbourhood of the relevant singular points. In order to do this we linearise equation (3.3.1) in the vicinity of each singular point in turn and then employ some results from the theory of ordinary differential equations.

An analysis of equation (3.3.1) near singular point C shows that any curve passing through C must be one of the following

$$Z = A(1-V)^{\frac{k(\gamma-1)}{\gamma(\sigma+1-k/\gamma)}} + \ldots\ldots ,$$

Singularity	Condition.	V	Z
A	$S = 0$, $Q = 0$, $D = 0$, $V \neq 1$.	$\frac{k}{\gamma\sigma} = V_A$.	$(1-V_A)^2 = Z$.
B	$S = 0$, $Q = 0$, $D \neq 0$.	$\frac{2}{[(\gamma+1)+\sigma(\gamma-1)]}$ $= V_B$.	$\frac{2(\sigma+1)^2}{\left\{\sigma+\left(\frac{\gamma+1}{\gamma-1}\right)\right\}\left\{2(\sigma+1)-\frac{k}{\gamma}[(\gamma+1)+\sigma(\gamma-1)]\right\}}$ $= Z_B$.
C	$S = 0$, $Q = 0$, $D = 0$, $V = 1$.	1 .	0 .
D	$\hat{Q} = 0$, $\hat{Z} = 0$.	$\frac{k}{\gamma(\sigma+1)} = V_D$.	∞ .
E	$\hat{S} = 0$, $V = 1$.	1 .	∞ .
H	$S = 0$, $V = 1$.	1 .	Arbitrary if $k = 0$.
I	$V = \infty$, $Z = \infty$.	∞ .	∞.
O	$V = 0$, $Z = 0$.	0 .	0 .

TABLE 1. Singular points in the phase plane.

$$Z = \left\{\frac{2-\sigma(\gamma-1)}{2(\sigma+1)-k(\gamma+1)/\gamma}\right\}(1-V)^2 + \ldots\ldots \quad , \qquad (3.3.6)$$

$$Z = B(1-V)^{\sigma(\gamma-1)} + \ldots\ldots \quad , \text{ for } \sigma(\gamma-1) > 2 \quad ,$$

where A, B are arbitrary, and the corresponding expressions for λ are

$$\lambda = \lambda^{*}\left\{1 + \frac{(1-V)}{(\sigma+1-k/\gamma)} + \ldots\ldots\right\} ,$$

$$\lambda = \lambda^{*}\left\{1 + \frac{(\gamma+1)(1-V)}{(\gamma-1)(\sigma+1)} + \ldots\ldots\right\} ,$$

$$\lambda = \lambda^{*}\left\{1 + (1-V) + \ldots\ldots\right\} ,$$

where λ^{*} must be chosen to satisfy boundary conditions. It is clear that, if $\lambda^{*} > 0$, λ increases as V decreases.

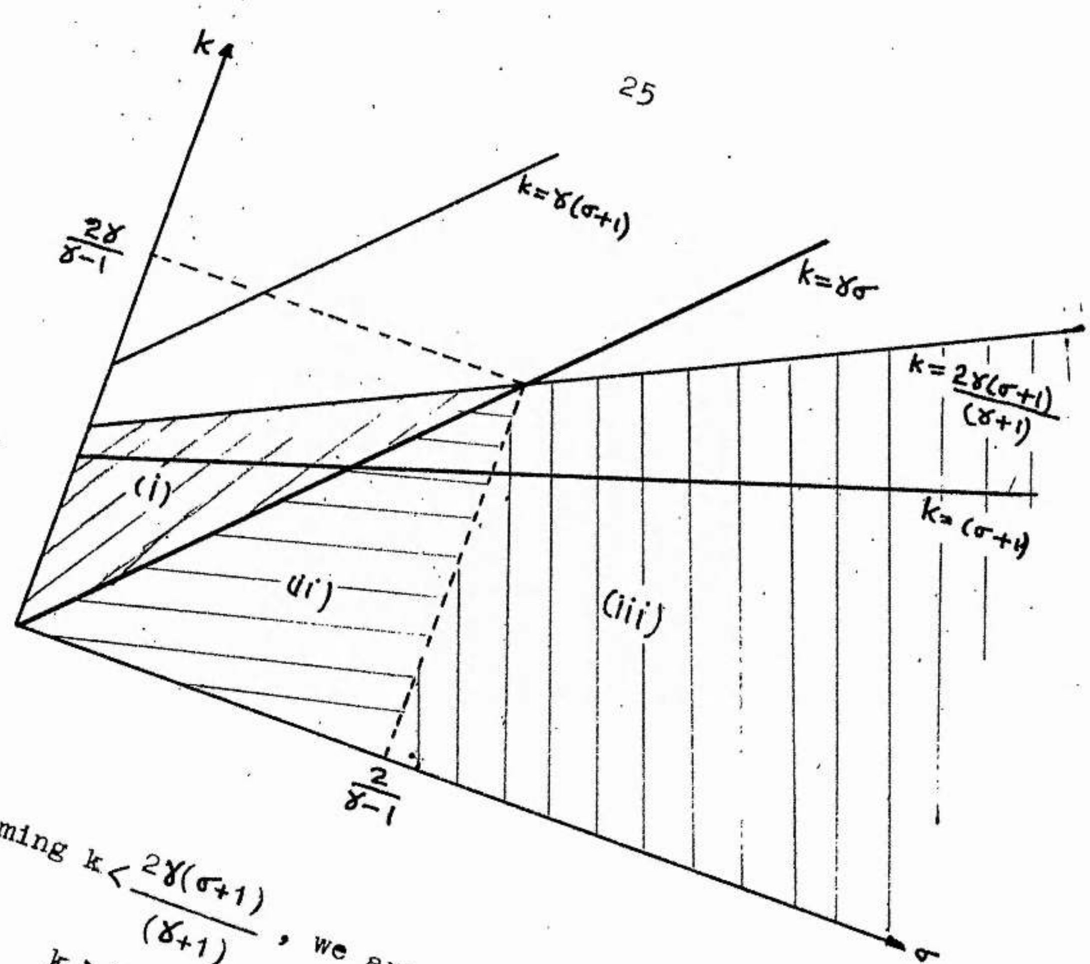
It appears that C is simply a node, but further investigation shows that the nature of this singular point depends further upon parameter values.

Near $Z = X = 0$, $X = (1-V)^2$,

$$\frac{dZ}{dX} = 0 \text{ along } Z = \frac{\gamma\sigma}{k} X \text{ , or } Z = 0 \text{ ,}$$

$$\frac{dZ}{dX} = \infty \text{ along } Z = \frac{X}{\sigma+1-k/\gamma} \text{ , or } X = 0 \text{ .}$$

Now looking at the k-σ parameter space,

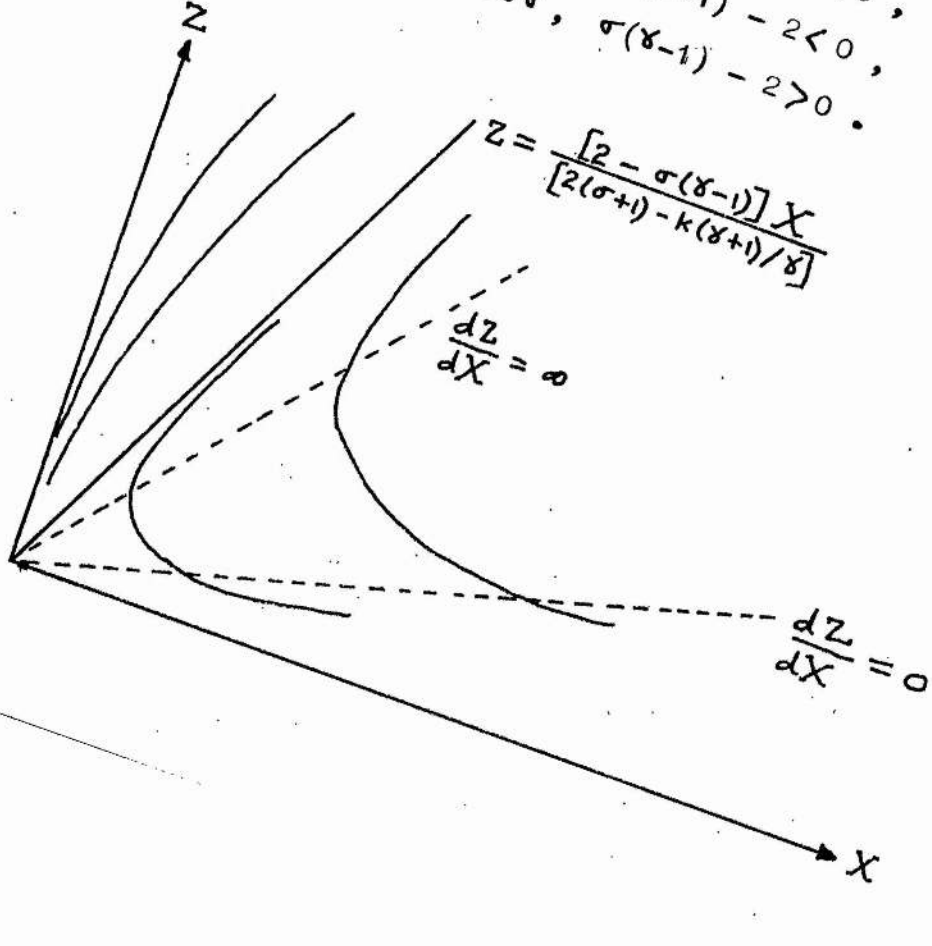


and assuming $k < \frac{2\gamma(\sigma+1)}{(\gamma+1)}$, we are interested in three regions

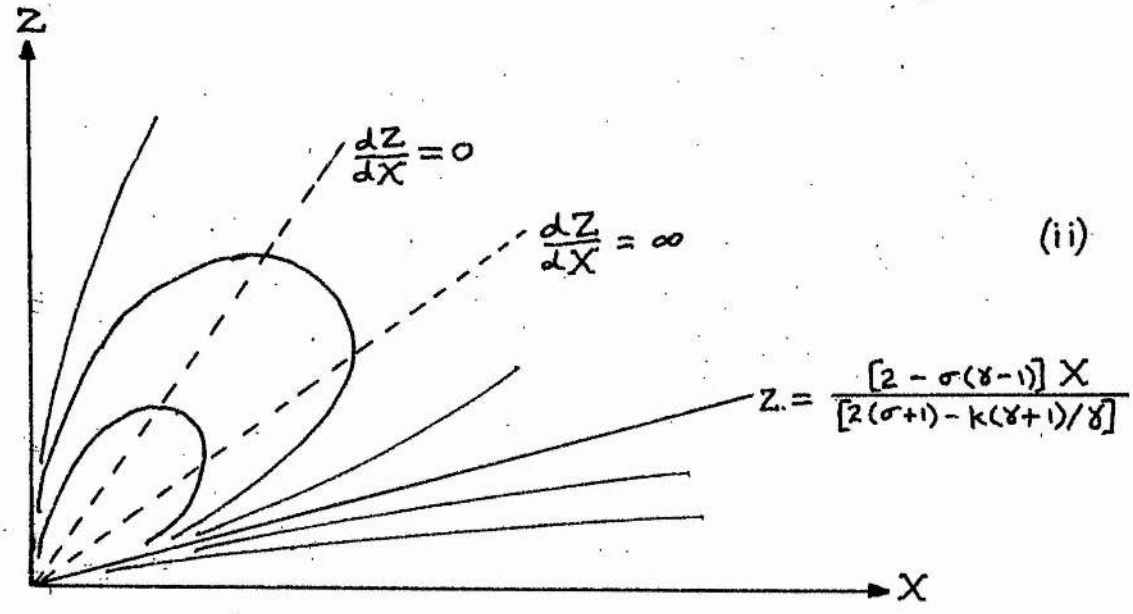
(i) $k > \gamma\sigma$, $\sigma(\gamma-1) - 2 < 0$,

(ii) $k < \gamma\sigma$, $\sigma(\gamma-1) - 2 < 0$,

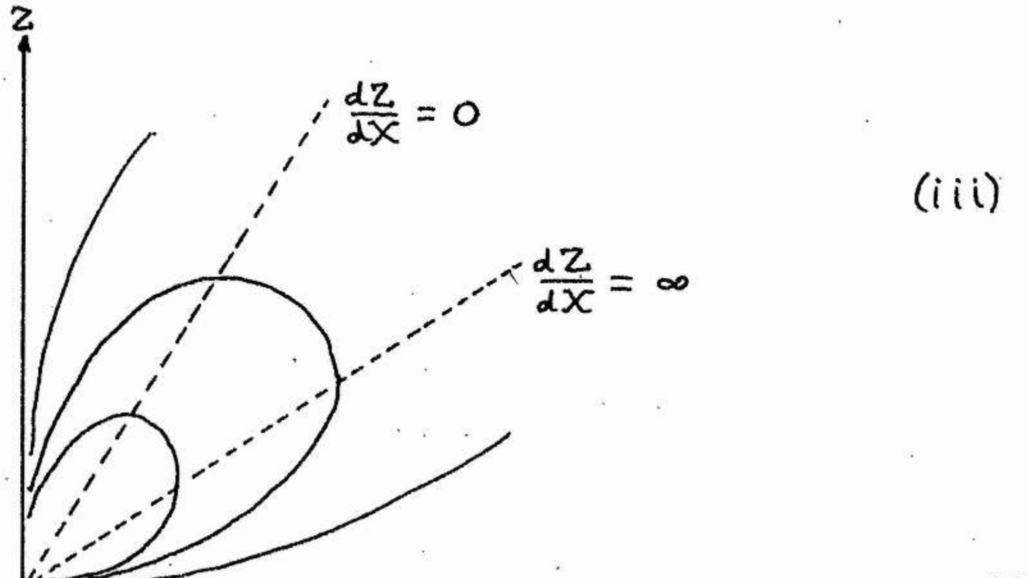
(iii) $k < \gamma\sigma$, $\sigma(\gamma-1) - 2 > 0$.



(i)



(ii)



(iii)

In region (i) of the k-σ parameter space $V_A > 1$ and singular point A is irrelevant, and also the integral curves below the one given by (3.3.6) exhibit saddle point behaviour as singular point C is approached. In regions (ii), (iii) of the k-σ parameter space, C is purely a node and all integral curves in the vicinity of C pass through C.

Near singular point A we define the small quantities x, y by

$$V = V_A + x \, , \qquad Z = Z_A + y \, . \tag{3.3.7}$$

Equation (3.3.1) now becomes approximately

$$\frac{dy}{dx} = \frac{Cx + Dy}{Ax + By} \, , \tag{3.3.8}$$

where
$$A = (1-V_A)\left\{(1-3V_A) - (\sigma+1)(1-V_A)\right\} \, ,$$
$$B = -V_A \, ,$$
$$C = -(1-V_A)^2\left\{4(1-V_A) + \sigma(\gamma-1)(1-3V_A)\right\} \, ,$$
$$D = -(1-V_A)\left\{2(1-V_A) - k(\gamma-1)/\gamma\right\} \, .$$

The solution of (3.3.8) is

$$(y + \alpha_1 x)^{\frac{1}{\beta_1}} = E(y + \alpha_2 x)^{\frac{1}{\beta_2}} \, , \tag{3.3.9}$$

where E is arbitrary and the α_i , β_i are the solutions of

$$B\alpha^2 - (D-A)\alpha - C = 0 \, ,$$

$$\beta^2 - (A+D)\beta + (AD-BC) = 0 \quad \text{respectively.}$$

From this we conclude that A is a saddle point for

$V_B = \dfrac{2}{(\gamma+1) + \sigma(\gamma-1)} < V_A < 1$, and then it is clear, from

(3.3.7) and (3.3.9) that the two integral curves

$$\left.\begin{aligned} Z &= Z_A - \alpha_1(V-V_A) + \ldots\ldots , \\ Z &= Z_A - \alpha_2(V-V_A) + \ldots\ldots , \end{aligned}\right\} \qquad (3.3.10)$$

will pass through A. Also λ takes a finite value and is monotonic on one of these two curves on passing through point A.

To explain the nature of singular point D we define the small quantities $y = \hat{Z}$, $x = V-V_D$, linearise equation (3.3.4) and then the local solution is

$$y^{-(\sigma+1)(1-V_D)}\left\{y - \left[(\sigma+1)+2(1-V_D)-k(\gamma-1)/\gamma\right]x\right\}^{\left[2(1-V_D)-k(\gamma-1)/\gamma\right]} = A \quad , \qquad (3.3.11)$$

A being arbitrary.

Since $0 < V_D < 1$, it follows from (3.3.11) that D is a node or a saddle point according as

$$k > \frac{2\gamma(\sigma+1)}{(\gamma+1) + \sigma(\gamma-1)} \quad ,$$

or $\quad k < \dfrac{2\gamma(\sigma+1)}{(\gamma+1) + \sigma(\gamma-1)}$ respectively.

As point D is approached along an integral curve, λ approaches infinity if D is a node or zero if D is a saddle point.

Near singular point B let $V = V_B + x$, $Z = Z_B + y$, x and y being small, and then equation (3.3.4) can be locally approximated by an equation similar to (3.3.8) but with different forms for A, B, C, D. It can readily be shown that point B is a node for $V_D < V_B < V_A$ and a saddle point for $V_B > V_A$.

For $V_B > V_A$, λ is infinite or zero at B depending upon which of the two integral curves we are on while, if $V_D < V_B < V_A$, λ approaches infinity along any curve moving towards B. For $V_B < V_D$ then $Z_B < 0$ and point B is irrelevant.

The other singular point of interest is point O and, for small Z, V an approximate solution of equation (3.3.1) is

$$V - \frac{k}{2\gamma} Z = A\sqrt{Z} \quad , \qquad (3.3.12)$$

where A is an arbitrary constant. Obviously the origin is a node and there is a curve of non-zero slope passing through it,

$$V = \frac{k}{2\gamma} Z + \ldots\ldots \quad , \text{ for } A = 0 \ , \qquad (3.3.13)$$

and a whole family of curves tangential to the V axis,

$$V = A\sqrt{Z} + \ldots\ldots \quad , \ 0 < |A| \leq \infty \ . \qquad (3.3.14)$$

The expressions for λ corresponding to (3.3.13), (3.3.14) are

$$\lambda = \lambda^{*} V^{-\frac{1}{2}} + \ldots\ldots \quad ,$$

$$\lambda = \lambda^{*} V^{-1} + \ldots\ldots \quad , \text{ respectively, where } \lambda^{*} \text{ is}$$

arbitrary. Obviously λ is infinite at O and this point corresponds to points at infinity in the gas for our problem.

The strong shock position, S, is a regular point for all γ, σ, k except for the particular case $\sigma = 0$, $k = 1$. It then coincides with singular point B and also

$$V_D = \gamma^{-1} < V_S = V_B$$

and thus λ at B is infinite.

The problem now is to construct an integral curve of (3.3.1) which passes through singular point C and the strong shock position S.

There is an analytic solution available when $k = (\sigma+1)$, the constant energy solution,

$$Z = \frac{(\gamma-1)\ V^2\ (1-V)}{2(V-\gamma^{-1})} \quad ,$$

$$\lambda = V^{-1}\left[\frac{\gamma V-1}{\gamma-1}\right]^{\frac{-1}{(\sigma-1)}}\left\{\frac{[2+(\gamma-1)(\sigma+1)]V-2}{(\sigma+1)(\gamma-1)}\right\}^{\frac{(\gamma+1)+\sigma^2(\gamma-1)}{[(\gamma+1)+\sigma(\gamma-1)](\sigma-1)}} ,$$

for $\sigma \neq 1$,

$$= V^{-1}\left[\frac{\gamma V-1}{\gamma-1}\right]^{\frac{(\gamma-1)}{2\gamma}} \mathrm{EXP}\left\{\frac{1}{2\gamma}\left[\left(\frac{\gamma-1}{\gamma V-1}\right)-1\right]\right\} \quad , \text{ for } \sigma = 1 .$$

It can be observed from (3.3.2) that, as an integral curve of (3.3.1) crosses the parabola $Z = (1-V)^2$ at any point other than the singular points A and C, the sign of $\frac{d(\ln\lambda)}{dV}$ changes. This indicates an extreme value of λ, Z and V would not be single valued functions of λ and hence this situation is physically unrealistic.

The ultimate objective in investigating these similarity solutions is to find the upper limit, $k_c(\gamma,\sigma)$, to k for which there is a solution to the problem under discussion.

There will be an integral curve connecting C to B or D, whichever corresponds to infinite λ, and also passing through A if $k < \gamma\sigma$, which bounds to the left, in the Z-V plane, all curves connecting C to B or D. This curve will be characterised by equation (3.3.6) if $k > \gamma\sigma$, or one of equations (3.3.10) if

$k < \gamma\sigma$ and we shall name it Γ for future reference.

It can be seen that any integral curve to the left of Γ for $\max(V_B, V_D) < V < \min(V_A, 1)$ cannot possibly reach C because of the saddle point behaviour of points A and C.

We are now in a position to sketch the integral curves as in FIGS. 3-6.

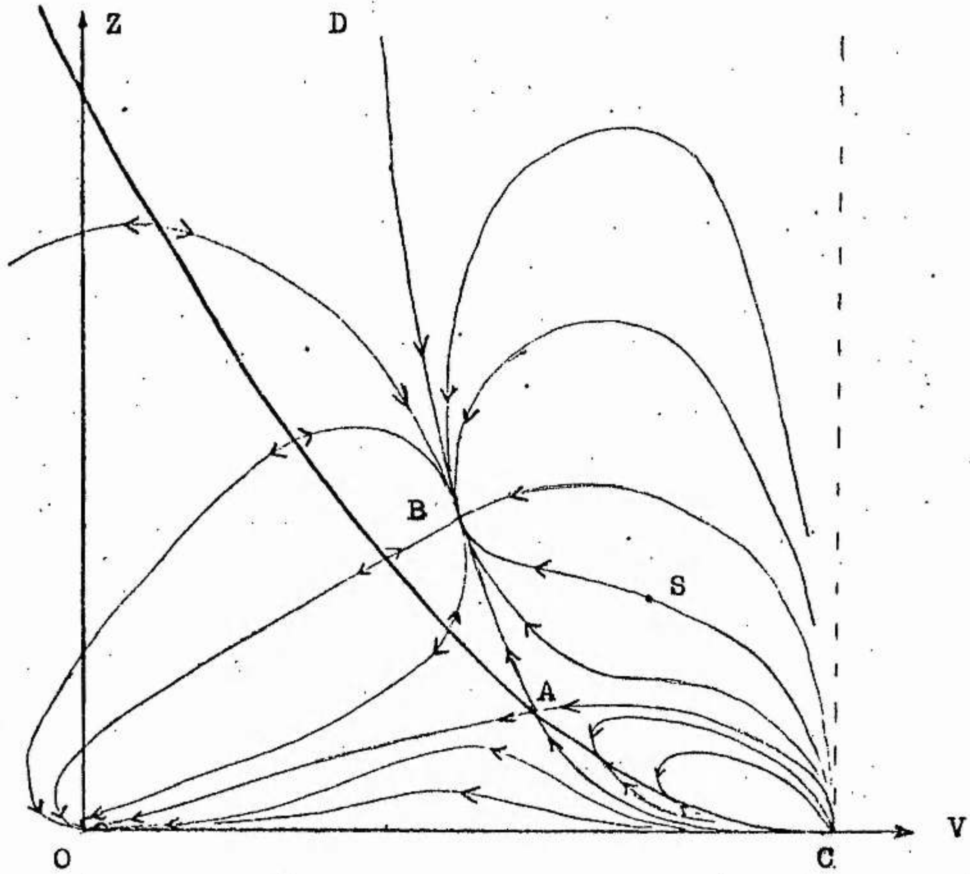


FIG. 3. Integral curves for $k < \gamma\sigma$, $Z_B > 0$.

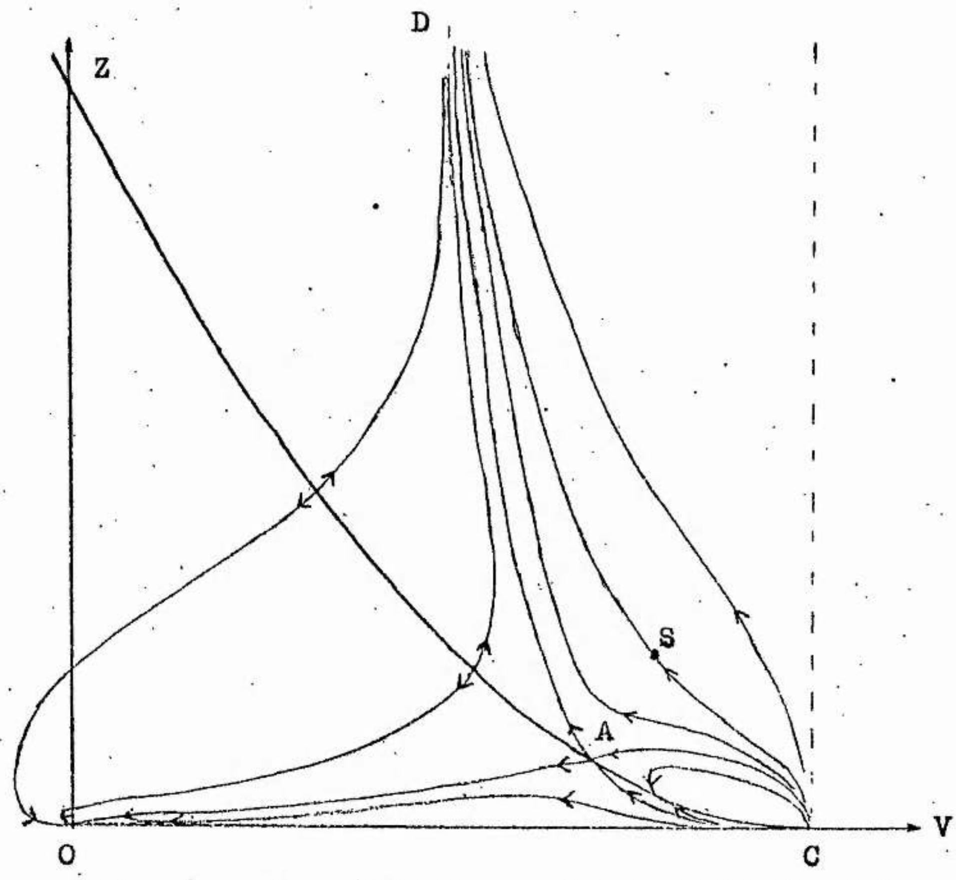


FIG. 4. Integral curves for $k < \gamma\sigma$, $Z_B < 0$.

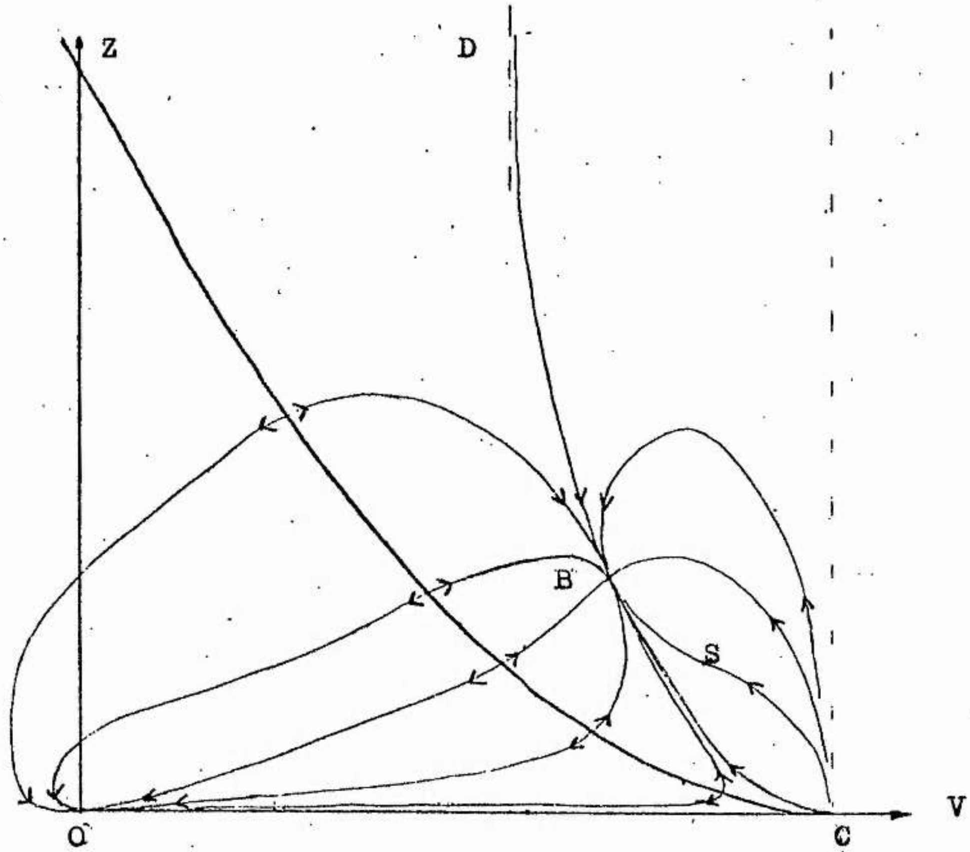


FIG. 5. Integral curves for $k > \gamma\sigma$, $Z_B > 0$.

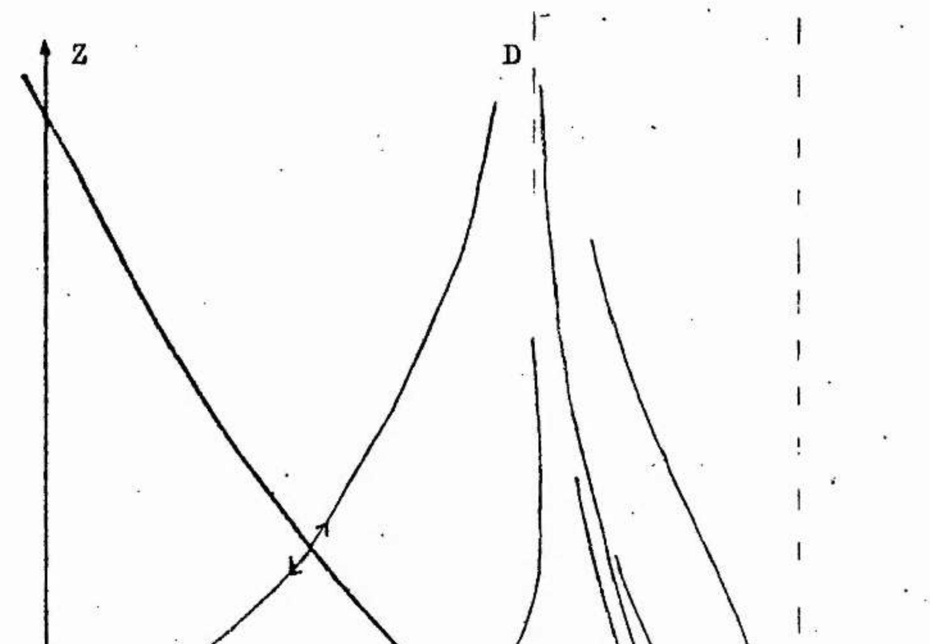


FIG. 6. Integral curves for $k > \gamma\sigma$, $Z_B < 0$.

3.4. Integral curves for $\delta > 1$.

In this section we study the integral curves of equation (3.2.3) with $\delta > 1$, $k > (\sigma+1)$. Neither Sedov nor Oppenheim et alia (1972) discussed this case and it is in this sense that the results displayed here are new.

Following section 3.3. we rewrite equations (3.2.3), (3.2.4) as

$$\frac{dZ}{dV} = \frac{Z\ S(V,Z)}{(1-V)\ Q(V,Z)} \quad , \tag{3.4.1}$$

$$\frac{d(\ln\lambda)}{dV} = \frac{D(V,Z)}{Q(V,Z)} \quad , \tag{3.4.2}$$

where it is clear how $D(V,Z)$, $Q(V,Z)$, $S(V,Z)$ are defined. Also if $\hat{Z} = Z^{-1}$ we have

$$\begin{aligned} \frac{d\hat{Z}}{dV} &= \frac{-\ \hat{Z}\ \hat{S}(V,\hat{Z})}{(1-V)\ \hat{Q}(V,\hat{Z})} \quad , \\ \frac{d(\ln\lambda)}{dV} &= \frac{\hat{D}(V,\hat{Z})}{\hat{Q}(V,\hat{Z})} \quad , \end{aligned} \tag{3.4.3}$$

where
$$\begin{aligned} \hat{D}(V,\hat{Z}) &= \hat{Z}\ D(V,Z) \quad , \\ \hat{Q}(V,\hat{Z}) &= \hat{Z}\ Q(V,Z) \quad , \\ \hat{S}(V,\hat{Z}) &= \hat{Z}\ S(V,Z) \quad . \end{aligned} \tag{3.4.4}$$

The singularities of this system are given in Oppenheim et alia (1972) , and a full list is included here in TABLE 2 for completeness.

An immediate remark is that singularity H can be ignored since its existence requires $\delta < 1$ for $k > 0$. With regard to singularities A and G it can quite easily be shown that they

Singularity	Condition.	V	Z
A	$D = 0$, $Q = 0$, $S = 0$.	$\frac{1}{2}\left[\frac{K}{\delta\sigma} + \frac{\delta-1}{\delta\sigma} + 1\right] + \left\{\frac{1}{4}\left[\frac{K}{\delta\sigma} + \frac{\delta-1}{\delta\sigma} + 1\right]^2 - \frac{K}{\delta\sigma}\right\}^{\frac{1}{2}} = V_A$.	$Z_A = (1-V_A)^2$.
G	$D = 0$, $Q = 0$, $S = 0$.	$\frac{1}{2}\left[\frac{K}{\delta\sigma} + \frac{\delta-1}{\delta\sigma} + 1\right] - \left\{\frac{1}{4}\left[\frac{K}{\delta\sigma} + \frac{\delta-1}{\delta\sigma} + 1\right]^2 - \frac{K}{\delta\sigma}\right\}^{\frac{1}{2}} = V_G$.	$Z_G = (1-V_G)^2$.
F	$Q = 0$, $Z = 0$.	$\frac{1}{\delta} = V_F$.	0 .
D	$\hat{Q} = 0$, $\hat{Z} = 0$.	$\frac{K}{\delta(\sigma+1)} = V_D$.	∞ .
C	$V = 1$, $Z = 0$.	1 .	0 .
B	$Q = 0$, $S = 0$, $D \neq 0$.	$\frac{2}{\delta\left[\sigma(\gamma-1)+(\gamma+1)\right]} = V_B$.	$Z_B = \frac{(\sigma+1)\gamma(\gamma-1)V_B^2(1-V_B)}{2\left[(\sigma-1)V_B + 2 - k\right]}$.
O	$V = 0$, $Z = 0$.	0 .	0 .
E	$\hat{S} = 0$, $V = 1$.	1 .	∞ .
I	$V = \infty$, $Z = \infty$.	∞ .	∞ .
H	$S = 0$, $V = 1$.	1 .	Arbitrary if $\delta = \frac{2}{2 + k(\gamma-1)}$.

TABLE 2. Singular points in the phase plane.

exist, that is V_A and V_G are both real, for $\delta > 1$ and further that $V_G < 1$, $V_A > 1$. Thus in the present discussion singular point A is irrelevant. Now we can investigate the behaviour of the integral curves of (3.4.1), (3.4.2) in the neighbourhood of the relevant singular points.

Near singular point F we have the approximate solution

$$V-\delta^{-1} = A\, Z^{\frac{1}{(\sigma+1)(\gamma-1)}} + \frac{(\sigma+1-K)\, Z}{(1-\delta^{-1})\left[(\sigma+1)(\gamma-1)-1\right]} + \ldots\ldots ,$$

$$\text{for} \quad (\sigma+1)(\gamma-1) \neq 1 ,$$

$$V-\delta^{-1} = A\, Z + \frac{(\sigma+1-K)}{(1-\delta^{-1})} Z \ln Z + \ldots\ldots ,$$

$$\text{for} \quad (\sigma+1)(\gamma-1) = 1 ,$$

where A is arbitrary. This clearly shows the nodal nature of singular point F. If $(\sigma+1)(\gamma-1) < 1$, then the integral curves consist of three types:

$$\left.\begin{aligned} & Z = 0 , \quad |A| = \infty , \\ & V-\delta^{-1} = \frac{(\sigma+1-K)\, Z}{(1-\delta^{-1})\left[(\sigma+1)(\gamma-1)-1\right]} + O\left(Z^{\frac{1}{(\sigma+1)(\gamma-1)}}\right), 0<|A|<\infty, \\ & V-\delta^{-1} = \frac{(\sigma+1-K)\, Z}{(1-\delta^{-1})\left[(\sigma+1)(\gamma-1)-1\right]} + o\left(Z^{\frac{1}{(\sigma+1)(\gamma-1)}}\right), \quad A = 0 . \end{aligned}\right\} \quad (3.4.5)$$

If $(\sigma+1)(\gamma-1) = 1$, then we have, correspondingly,

$$\left.\begin{aligned} & Z = 0 , \quad |A| = \infty , \\ & V-\delta^{-1} = \frac{(\sigma+1-K)}{(1-\delta^{-1})} Z \ln Z + O(Z) , \quad 0<|A|<\infty , \\ & V-\delta^{-1} = \frac{(\sigma+1-K)}{(1-\delta^{-1})} Z \ln Z + o(Z) , \quad A = 0 , \end{aligned}\right\} \quad (3.4.6)$$

while for $(\sigma+1)(\gamma-1) > 1$

$$\left.\begin{array}{l} Z = 0 \ , \ |A| = \infty \ , \\[2ex] V-\delta^{-1} = A \, Z^{\frac{1}{(\sigma+1)(\gamma-1)}} + O(Z) \ , \ 0 < |A| < \infty \ , \\[2ex] V-\delta^{-1} = \dfrac{(\sigma+1-K)\, Z}{(1-\delta^{-1})\left[(\sigma+1)(\gamma-1)-1\right]} + o(Z) \ , \ A = 0 \ . \end{array}\right\} \quad (3.4.7)$$

The first of each of these three sets of equations is a trivial solution of (3.4.1), the second describes a whole family of curves while the third represents a particular curve passing through F.

The corresponding equations for λ are

$$\lambda \ = \lambda^{*}(V- \ ^{-1})^{(\ -1)} + \ldots\ldots \ , \ |A| = \infty \ ,$$

$$\lambda \ = \lambda^{*} \, Z^{\frac{(\ -1)}{(\ +1)(\ -1)}} + \ldots\ldots \ , \ 0 \leq |A| < \infty \ ,$$

where λ^{*} is arbitrary. It is clear that $\lambda \to 0$ as F is approached along any integral curve.

Singular point C could correspond to some streamline in the flow field and a study of the integral curves near it can be made by letting $x = V-1$, Z be small quantities. We find that C is a node and the integral curves are given approximately by

$$Z = \frac{-\gamma(1-\delta^{-1})}{(k-\sigma-1)}\, x + o(x^{a}) \ ,$$

$$\left.\begin{aligned} Z &= \frac{-\gamma(1-\delta^{-1})}{(k-\sigma-1)}\, x + O(x^a)\ , \\ Z &= 0\ , \\ x &= 0\ . \end{aligned}\right\} \qquad (3.4.8)$$

where $a = \dfrac{2(1-\delta^{-1})+k\gamma-(\sigma+1)}{2(1-\delta^{-1})+(\gamma-1)(\sigma+1)}$.

The similarity variable, λ, takes a finite value at C and the expressions for λ corresponding to (3.4.8) are, for arbitrary λ^*,

$$\lambda = \lambda^*\left\{1 - \frac{\gamma\, x}{(\gamma-1)(\sigma+1)+2(1-\delta^{-1})} + \ldots\ldots\right\} ,$$

for the first two, and

$$\lambda = \lambda^*\left\{1 + \frac{x^2}{2(1-\delta^{-1})} + \ldots\ldots\right\} \quad \text{, for the third.}$$

Near singularity G if we define the small quantities $x = V-V_G$, $y = Z-Z_G$, then we can approximate equation (3.4.1) by one similar to (3.3.8) but with different expressions for A, B, C, D . The general solution of this approximate equation is then similar to (3.3.9). If $(AD-BC) = \beta_1\beta_2 < 0$ then G is a saddle point, otherwise it is a node. In this case the quantity (AD-BC) does not lend itself easily to examination of sign for general parameter values although for $K = (\sigma+1)$ it is definitely negative. Even though we do not prove analytically that G is always a saddle point we can say, from geometrical considerations, that it must be otherwise there would have to be an isolated point somewhere in the curvilinear triangle CGF. The value of λ at G is finite and changes monotonically on an integral curve passing through G.

The nature of singular point D can be found in a similar way to that of the corresponding point in the previous section. We find that, for $0 < V_D < 1$, D is a saddle point for

$$\left.\begin{array}{l} K < \dfrac{2(\sigma+1)}{2+(\gamma-1)(\sigma+1)} \ , \\[2ex] \text{or a node for} \\[2ex] K > \dfrac{2(\sigma+1)}{2+(\gamma-1)(\sigma+1)} \ , \end{array}\right\} \qquad (3.4.9)$$

the respective integral curves being of the form

$$Z^{-1} = \left\{\frac{2(V_D-\delta^{-1})+K(\gamma-1)\delta^{-1}-(\sigma+1)(1-V_D)}{V_D(V_D-\delta^{-1})(1-V_D)^2}\right\}(V-V_D)+\ldots\ldots \quad ,$$

$$Z^{-1} = O((V-V_D)^a) \ ,$$

with $a = \dfrac{(\sigma+1)(1-V_D)}{2(V_D-\delta^{-1})+K(\gamma-1)\delta^{-1}}$.

If D is a saddle point then it corresponds to zero λ, while if it is a node then it corresponds to infinite λ.

We are only interested in singular point B for $K < \dfrac{2(\sigma+1)}{2+(\gamma-1)(\sigma+1)}$, that is $Z_B > 0$, and then it can be shown that B is a node, λ approaching infinity as B is approached along an integral curve. There will be two types of integral curves passing through B and these will be given approximately by an equation similar to equation (3.3.9).

We can see that points B and D are, in a sense, complementary but we disregard B for $Z_B < 0$.

The final singularity of interest is point O, the origin, and it becomes clear that it is a node. Any integral curve in its vicinity is given by one of the following

$$V = KZ + o(Z) \quad ,$$

$$V = AZ^{\frac{1}{2}} + O(Z) \quad ,$$

$$Z = 0 \quad ,$$

where A is arbitrary, and the corresponding expressions for λ are

$$\lambda = \lambda^{*} V^{-\frac{1}{2}\delta} + \ldots\ldots \quad ,$$

$$\lambda = \lambda^{*} V^{-\delta} + \ldots\ldots \quad ,$$

$$\lambda = \lambda^{*} V^{-\delta} + \ldots\ldots \quad .$$

As in the previous section point O corresponds to points at infinity in physical space.

When $K = (\sigma+1)$ there is an analytic solution for the integral curve of (3.4.1) which connects F to D and passes through G. This curve is the vertical line $V = \delta^{-1}$, for all

Z. For this value of K, $Z_B < 0$. For other values of K there are no analytic solutions available, but we are in a position to sketch the integral curves for $K < (\sigma+1)$, $K = (\sigma+1)$, $K > (\sigma+1)$. These curves are depicted in FIGS. 7-12. It is observed from these sketches that there is always an integral curve connecting F to D through G, which we shall call Δ for future reference, and as this curve is traversed through G λ changes monotonically. If $(\gamma-1)(\sigma+1) < 1$, then Δ is characterised by the second of (3.4.5), for $(\gamma-1)(\sigma+1) = 1$ the second of (3.4.6) gives the local behaviour while, for $(\gamma-1)(\sigma+1) > 1$, the second of (3.4.7) is the relevant solution if A is chosen suitably. It is clear that, for $(\gamma-1)(\sigma+1) \leq 1$, Δ is explicitly defined near F, whereas, if $(\gamma-1)(\sigma+1) > 1$, there is some degree of indeterminacy, namely the coefficient A in the second of equations (3.4.7). That is not to say that we cannot find A, since we know the path of Δ as it leaves G towards F and we can construct the curve, numerically if necessary, from G to F and hence compute the coefficient A as accurately as possible. Even so, there is a marked difference in the solutions for $(\gamma-1)(\sigma+1) \leq 1$ and $(\gamma-1)(\sigma+1) > 1$. As $(\gamma-1)(\sigma+1)$ approaches unity from below, Δ becomes less inclined to the V axis near F until, at $(\gamma-1)(\sigma+1) = 1$, the curve leaves F tangentially to the V axis and does so for $(\gamma-1)(\sigma+1) > 1$.

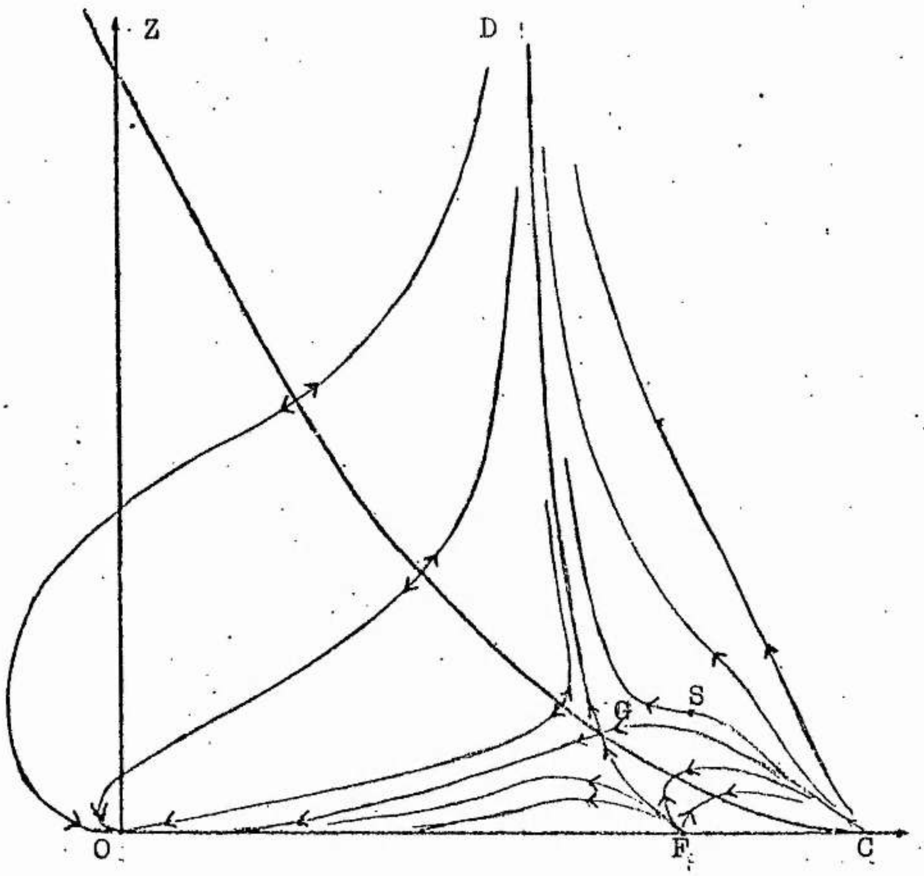


FIG. 7. Integral curves for K < $(\sigma+1)$, $(\gamma-1)(\sigma+1) < 1$.

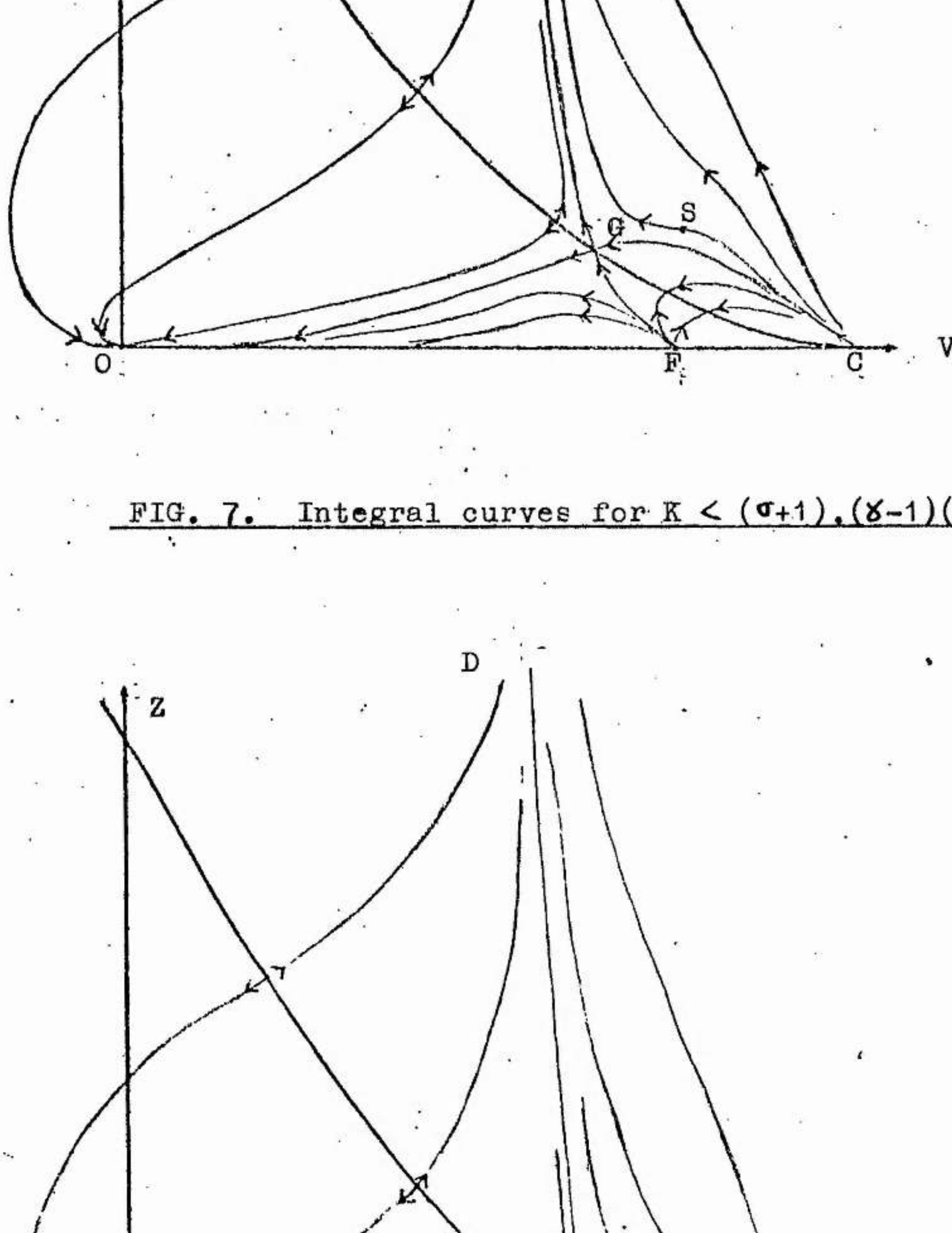


FIG. 8. Integral curves for K < $(\sigma+1)$, $(\gamma-1)(\sigma+1) \geq 1$.

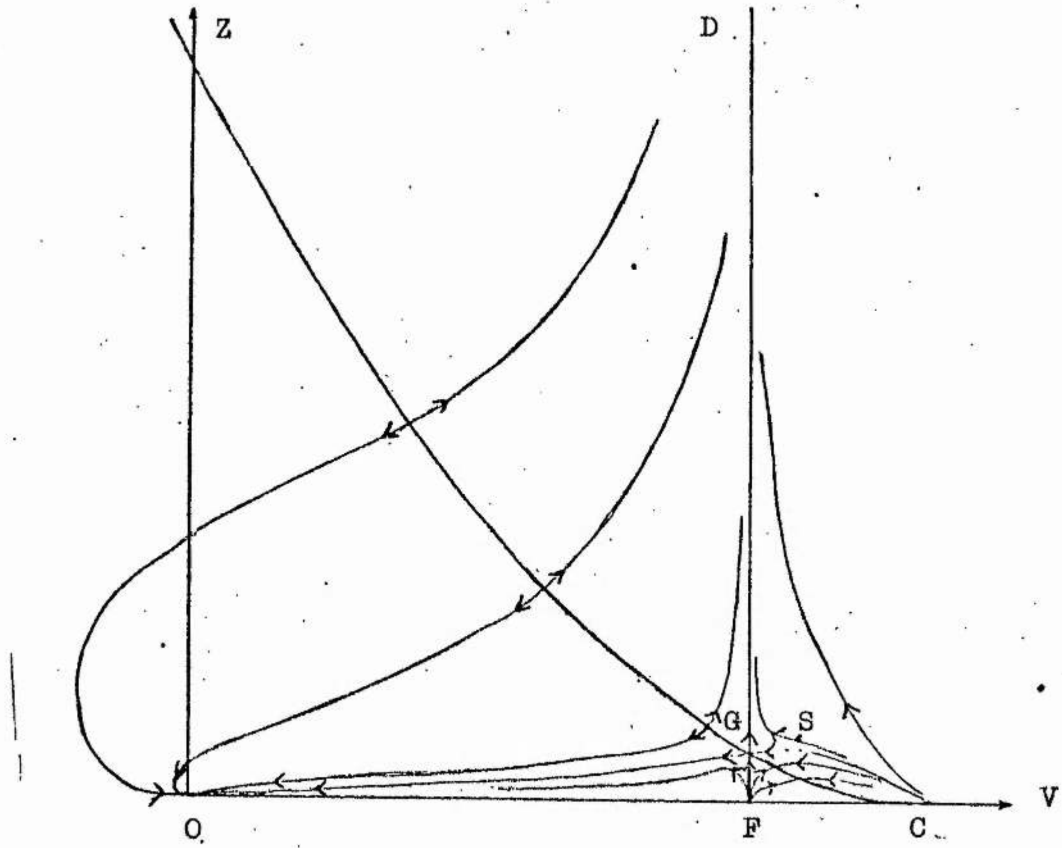


FIG. 9. Integral curves for $K = (\sigma+1), (\gamma-1)(\sigma+1) < 1$.

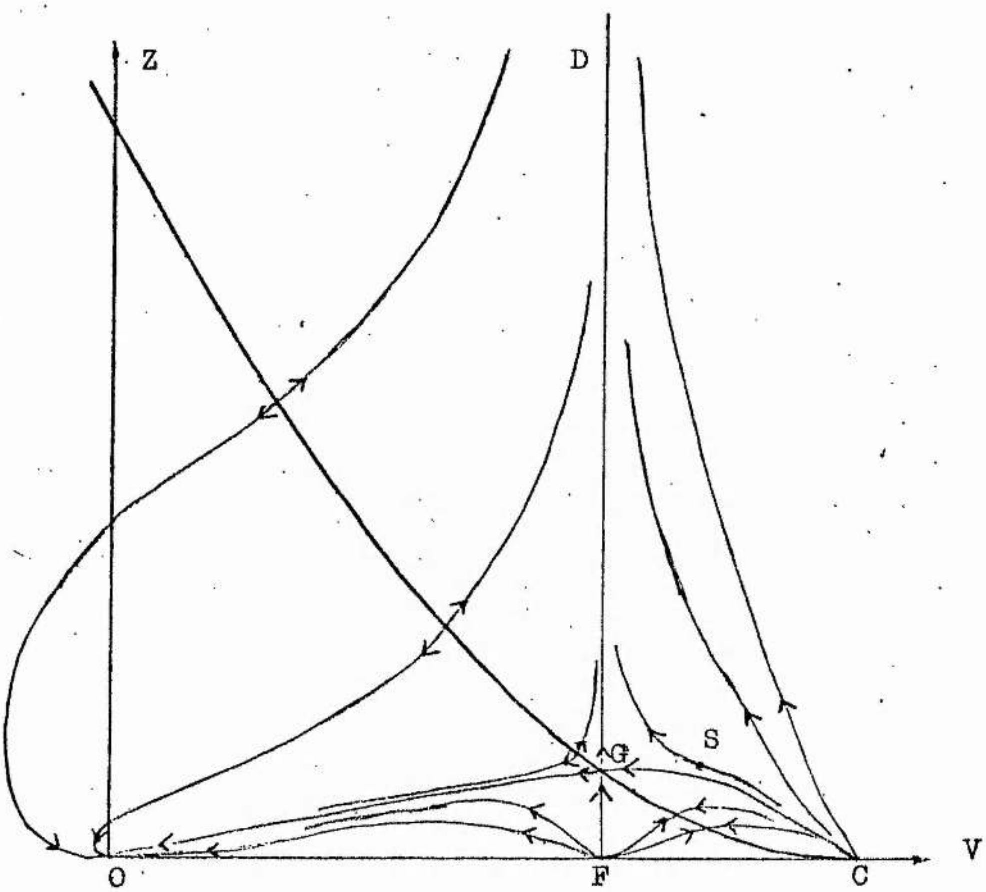


FIG. 10. Integral curves for $K = (\sigma+1), (\gamma-1)(\sigma+1) \geq 1$.

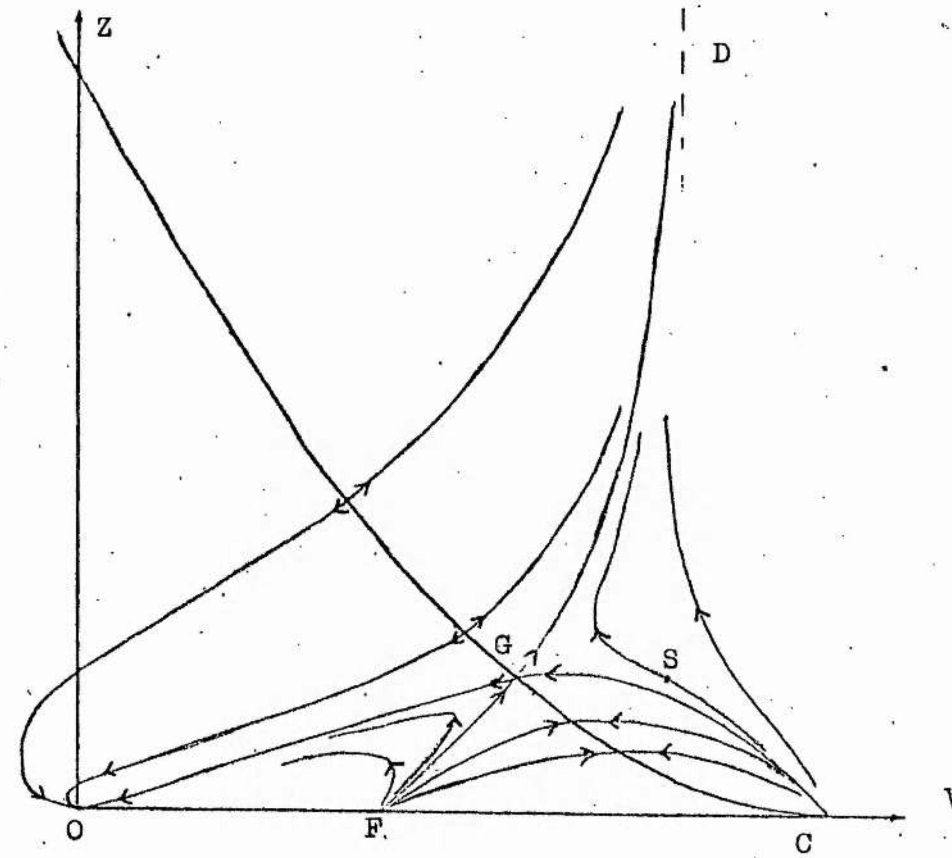


FIG. 11. Integral curves for K $> (\sigma+1), (\gamma-1)(\sigma+1) < 1$.

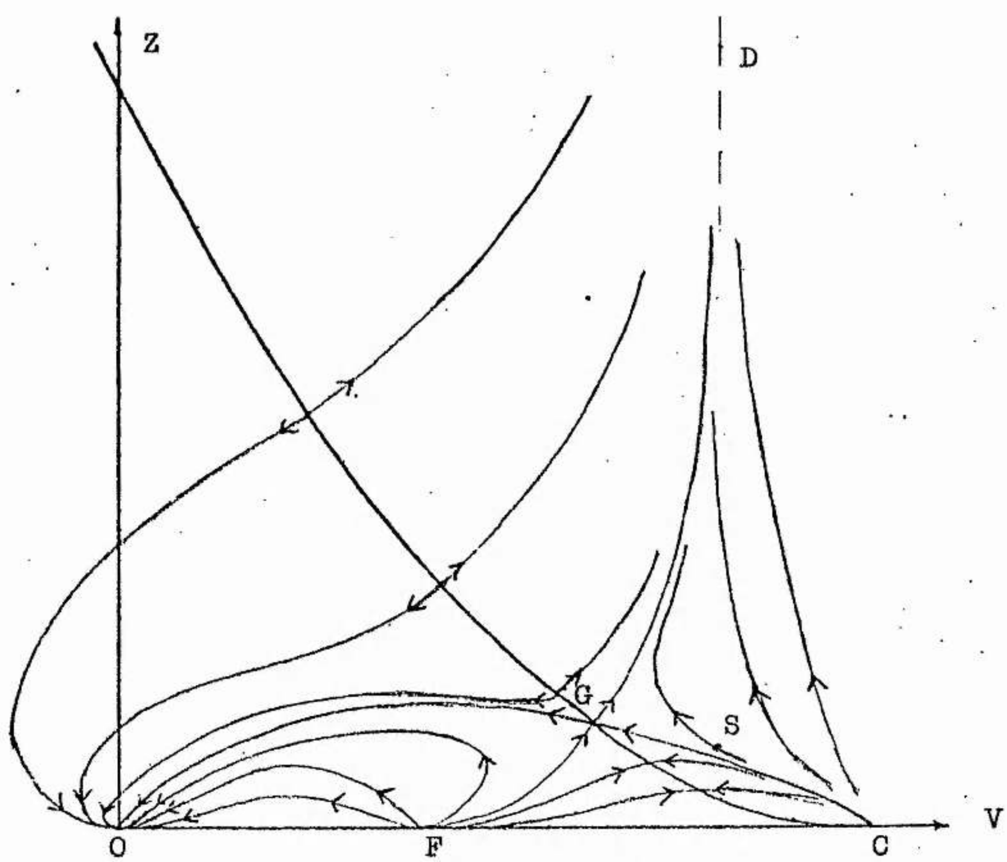


FIG. 12. Integral curves for K $> (\sigma+1), (\gamma-1)(\sigma+1) \geqslant 1$.

4. The Lagrangian formulation.

4.1. Introduction.

Although we use similarity solutions to display certain features of the zeroth order inner problem it was decided to solve the problem using the Lagrangian formulation of the partial differential equations. The reason for this decision was due to the relative ease with which the matching of the inner and outer expansions could be carried out. In this chapter we derive the outer and inner equations in the Lagrangian formulation and restate the boundary conditions.

4.2. Definition of particle path function, outer equations and boundary conditions.

From the first of equations (2.3.6) we define the particle path function, ψ, as

$$\left(\frac{\partial\psi}{\partial r}\right)_t = \rho r^\sigma \quad , \quad \left(\frac{\partial\psi}{\partial t}\right)_r = -\rho u r^\sigma \ . \qquad (4.2.1)$$

Physically, ψ may be regarded as a measure of the mass of gas between a general and some reference particle path in the flow field; here we take the contact front as that reference.

FIG. 13. depicts the particle paths, lines of constant ψ, in the r-t plane ahead of the contact front.

The shock path in the r-t plane is defined by the equation,

$$t = F(r) \ ,$$

and to obtain the corresponding equation in the ψ-r plane we note that, when the shock is at $(r,F(r))$ in the r-t plane, the mass of gas between the shock and the contact front is exactly the same as the mass of gas initially detween radii r and unity.

Thus integrating the first of equations (4.2.1) between 1 and r, at t = 0 , gives the shock path in the ψ-r plane, that is

$$\psi = \int_1^r r^{\sigma-k}\,dr \quad ,$$

since $\rho = r^{-k}$ for $r > 1$, $t = 0$.

Carrying out this integration we obtain

$$1 + (\sigma+1-k)\psi = r^{\sigma+1-k} \,, \quad k \neq (\sigma+1) \quad ,$$

$$\psi = \ln r \quad , \quad k = (\sigma+1) \quad .$$

It will simplify future analysis if we introduce, for $k \neq (\sigma+1)$, ϕ , where

$$\phi = 1 + (\sigma+1-k)\psi \quad . \qquad (4.2.2)$$

Introducing ϕ for $k \neq (\sigma+1)$, or ψ for $k = (\sigma+1)$, therefore, at the expense of t in (2.3.6) gives

$$\left.\begin{aligned} & u\frac{\partial\rho}{\partial r} + \rho\frac{\partial u}{\partial r} + (\sigma+1-k)\rho^2 r^{\sigma}\frac{\partial u}{\partial\phi} + \frac{\sigma\rho u}{r} = 0 \quad , \\ & u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + (\sigma+1-k)r^{\sigma}\frac{\partial p}{\partial\phi} = 0 \quad , \\ & u\frac{\partial}{\partial r}(p/\rho^{\gamma}) = 0 \quad , \end{aligned}\right\} \qquad (4.2.3)$$

for $k \neq (\sigma+1)$, or

$$\left.\begin{aligned} &u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \rho^2 r^{\sigma}\frac{\partial u}{\partial \psi} + \frac{\sigma\rho u}{r} = 0 \quad , \\ &u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + r^{\sigma}\frac{\partial p}{\partial \psi} = 0 \quad , \\ &u\frac{\partial}{\partial r}(p/\rho^{\gamma}) = 0 \quad , \end{aligned}\right\} \tag{4.2.4}$$

for $k = (\sigma+1)$.

The contact front is located by

$$\psi = 0 \quad \text{or} \quad \phi = 1 \quad , \quad \text{where} \quad u = 1 \quad . \tag{4.2.5}$$

Also the shock lies on the curve

$$\left.\begin{aligned} \phi &= r^{\sigma+1-k} \quad , \ k \neq (\sigma+1) \quad , \\ \psi &= \ln r \quad , \ k = (\sigma+1) \quad , \end{aligned}\right\} \tag{4.2.6}$$

on which

$$\left.\begin{aligned} u &= \frac{2V_1(r)}{(\gamma+1)} \quad , \\ p &= \frac{2V_1^2(r)}{(\gamma+1)} r^{-k} \quad , \\ \rho &= \frac{(\gamma+1)}{(\gamma-1)} r^{-k} \quad . \end{aligned}\right\} \tag{4.2.7}$$

The third of equations (4.2.3) or (4.2.4) means that the quantity p/ρ^{γ} is a function of ϕ or ψ only and we can determine this function by its value on the shock. Thus we have

$$p/\rho^{\gamma} = \frac{2}{(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} V_1^2(x)\ x^{k(\gamma-1)} \quad , \tag{4.2.8}$$

where $$x = \begin{cases} \phi^{\frac{1}{\sigma+1-k}} & , k \neq (\sigma+1) \ , \\ e^{\psi} & , k = (\sigma+1) \ . \end{cases} \tag{4.2.9}$$

The equations, then, are the first two equations of (4.2.3) or (4.2.4) together with (4.2.8) and (4.2.9). The boundary conditions are given by (4.2.5), (4.2.6) and (4.2.7). In our asymptotic analysis we will call ϕ, or ψ, the outer variable, and call (4.2.3) and (4.2.4) with (4.2.8), (4.2.9) the outer equations.

4.3. The inner equations.

We define the inner variable, ϕ_1, by

$$\phi_1 = \begin{cases} \phi \, r^{k-\sigma-1} & , k \neq (\sigma+1) \ , \\ e^{\psi} r^{-1} & , k = (\sigma+1) \ . \end{cases} \tag{4.3.1}$$

Introducing ϕ_1 at the expense of ϕ or ψ in the outer equations yields the inner equations:

$$\left.\begin{aligned} & ur\frac{\partial \rho}{\partial r} - (\sigma+1-k)\,\phi_1 u\frac{\partial \rho}{\partial \phi_1} + \rho r\frac{\partial u}{\partial r} - (\sigma+1-k)\phi_1\rho\frac{\partial u}{\partial \phi_1} \\ & + (\sigma+1-k)\rho^2 r^k\frac{\partial u}{\partial \phi_1} + \sigma\rho u = 0 \ , \\ & \rho ur\frac{\partial u}{\partial r} - (\sigma+1-k)\rho\phi_1\frac{\partial u}{\partial \phi_1} + r\frac{\partial p}{\partial r} - (\sigma+1-k)\phi_1\frac{\partial p}{\partial \phi_1} \\ & + (\sigma+1-k)\rho r^k\frac{\partial p}{\partial \phi_1} = 0 \ , \\ & p\rho^{-\gamma} = \frac{2}{(\gamma+1)}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} V_1^2\left(\phi_1^{\frac{1}{\sigma+1-k}}\, r\right)\phi_1^{\frac{k(\gamma-1)}{\sigma+1-k}}\, r^{k(\gamma-1)} , \end{aligned}\right\} \tag{4.3.2}$$

for $k \neq (\sigma+1)$,

$$ur \frac{\partial \rho}{\partial r} - \phi_1 u \frac{\partial \rho}{\partial \phi_1} + \rho r \frac{\partial u}{\partial r} - \phi_1 \rho \frac{\partial u}{\partial \phi_1}$$
$$+ \rho^2 r^k \phi_1 \frac{\partial u}{\partial \phi_1} + \sigma \rho u = 0 \quad ,$$

$$\rho u r \frac{\partial u}{\partial r} - \phi_1 \rho u \frac{\partial u}{\partial \phi_1} + r \frac{\partial p}{\partial r} - \phi_1 \frac{\partial p}{\partial \phi_1} \qquad (4.3.3)$$
$$+ \rho r^k \phi_1 \frac{\partial p}{\partial \phi_1} = 0 \quad ,$$

$$p\rho^{-\gamma} \doteq \frac{2}{(\gamma+1)} \left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} V_1^2(\phi_1 r)\ \phi_1^{k(\gamma-1)}\ r^{k(\gamma-1)} \quad ,$$

for $k = (\sigma+1)$.

These, then, are the inner equations and are used to produce an asymptotic solution valid near the shock, which is now located by $\phi_1 = 1$.

The zeroth order inner solution of these equations is exactly the same as the corresponding similarity solution. To obtain λ as a function of ϕ_1 we use the definitions of ψ, ϕ and then ϕ_1 to give

$$\lambda = \frac{b_0\left(\frac{\gamma+1}{\gamma-1}\right) \rho_0(\phi_1)\ U_0(\phi_1)}{\left(\frac{\gamma+1}{\gamma-1}\right)\rho_0(\phi_1) - f(\phi_1)} \quad , \qquad (4.3.4)$$

where $f(\phi_1) = \begin{cases} \phi_1 & \text{for } k \neq (\sigma+1) \quad , \\ 1 & \text{for } k = (\sigma+1) \quad , \end{cases}$

and the functions $\rho_0(\phi_1)$, $b_0U_0(\phi_1)$, which are also used in later chapters, are the asymptotic forms of $\frac{(\gamma-1)}{(\gamma+1)}\, r^{-k}\rho$, $r^{-\epsilon}\, u$ respectively.

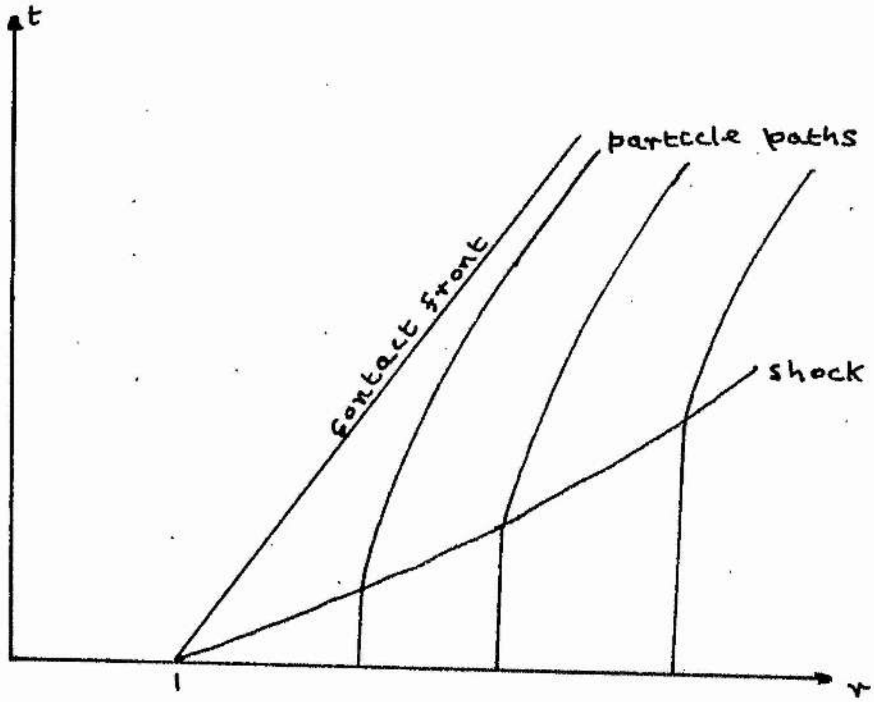


FIG. 13.

5. Matched asymptotic expansions for $k < k_c$.

5.1. The outer expansions.

In this section we seek asymptotic solutions of the outer equations with the boundary condition on the contact front. We find that the form of these solutions depends upon parameter values.

The relevant equations are those of (4.2.3), (4.2.4), (4.2.8), (4.2.9) and, in seeking asymptotic solutions of these, we assume expansions of the form, for $k < (\sigma+1)$,

$$\left.\begin{aligned} u &= 1 + \frac{u_1(\phi,r)}{r^{\sigma+1-k/\gamma}} + \ldots\ldots \quad , \\ p &= \frac{(\gamma+1)}{2}\, r^{-k}\left\{\Pi_0(r) + \frac{\Pi_1(\phi,r)}{r^{\sigma+1-k/\gamma}} + \ldots\ldots\right\} \quad , \\ \rho &= \frac{(\gamma+1)}{(\gamma-1)}\, r^{-k/\gamma}\left\{R_0(\phi,r) + \frac{R_1(\phi,r)}{r^{\sigma+1-k/\gamma}} + \ldots\ldots\right\} \quad , \end{aligned}\right\} \tag{5.1.1}$$

where

$$\frac{\partial u_1(\phi,r)}{\partial \phi} = \frac{-(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k)(\gamma+1)R_0(\phi,r)} \quad , \tag{5.1.2}$$

$$\frac{\partial \Pi_1(\phi,r)}{\partial \phi} = \frac{(\gamma-1)\left[k\Pi_0(r) - r\dfrac{d\Pi_0(r)}{dr}\right]}{(\sigma+1-k)\;(\gamma+1)\;\;R_0(\phi,r)} \quad , \tag{5.1.3}$$

$$R_0(\phi,r) = \left\{\frac{\Pi_0(r)}{V^2\left(\phi^{\frac{1}{\sigma+1-k}}\right)}\right\}^{-1} \phi^{\frac{-k(\gamma-1)}{(\sigma+1-k)}} \quad , \tag{5.1.4}$$

$$R_1(\phi,r) = \frac{R_0(\phi,r)\,\Pi_1(\phi,r)}{\gamma\,\Pi_0(r)} \quad , \tag{5.1.5}$$

for $k = (\sigma+1)$, (5.1.1) with ψ replacing ϕ together with

$$\frac{\partial u_1(\psi,r)}{\partial\psi} = \frac{-(\sigma-k/\gamma)\ (\gamma-1)}{(\gamma+1)\ R_0(\psi,r)} \quad , \tag{5.1.6}$$

$$\frac{\partial\Pi_1(\psi,r)}{\partial\psi} = \frac{2k(\gamma-1)u_1(\psi,r)}{\gamma(\gamma+1)} + \frac{(\gamma-1)\left[k\Pi_0(r) - r\,\frac{d\Pi_0(r)}{dr}\right]}{(\gamma+1)\ R_0(\psi,r)} \quad , \tag{5.1.7}$$

$$R_0(\psi,r) = \left\{\frac{\Pi_0(r)}{V^2(\ e^{\psi})}\right\}^{-1} e^{-k(\gamma-1)\psi/\gamma} \quad , \tag{5.1.8}$$

$$R_1(\psi,r) = \frac{R_0(\psi,r)\,\Pi_1(\psi,r)}{\gamma\,\Pi_0(r)} \quad , \tag{5.1.9}$$

and, for $k > (\sigma+1)$,

$$\left.\begin{aligned}
u &= 1 + \frac{u_1(\phi,r)}{r^{\sigma+1-k/}} + \cdots\cdots \quad , \\
p &= \frac{(\gamma+1)}{2}\, r^{-k}\left\{\Pi_0(r) + \frac{\Pi_1(\phi,r)}{r^{2(\sigma+1)-k(\gamma+1)/\gamma}} + \cdots\right\}, \\
\rho &= \frac{(\gamma+1)}{(\gamma-1)}\, r^{-k/\gamma}\left\{\ R_0(\phi,r)\right. \\
&\qquad + \frac{R_1(\phi,r)}{r^{2(\sigma+1)-k(\gamma+1)/\gamma}} + \cdots\cdots \quad ,
\end{aligned}\right\} \tag{5.1.10}$$

where (5.1.2), (5.1.4), (5.1.5) hold but (5.1.3) is replaced by

$$\frac{\partial \Pi_1(\phi,r)}{\partial \phi} = \frac{2(\sigma+1-k/\gamma)\, u_1(\phi,r)}{(\gamma+1)\,(\sigma+1-k)} \quad . \qquad (5.1.11)$$

For all k we have

$$\Pi_0(r) = \alpha_0 F_0(r) = \alpha_0 \left\{ 1 + \sum_{i=1}^{\infty} f_i(r) \right\} ,$$

where $O(1) > f_i(r) > \max\left\{O(r^{-(\sigma+1-k/\gamma)}), O(r^{k(\gamma+1)/\gamma-2(\sigma+1)})\right\}$, for all i, as $r \to \infty$. Also we have put

$$V_1(r) = \frac{(\gamma+1)}{2} V(r) \quad .$$

It can be seen from equation (5.1.3) that, if $F_0(r) = 1$, $k = 0$ is a special case. In these circumstances equation (5.1.3) becomes $\dfrac{\partial \Pi_1(\phi,r)}{\partial \phi} = 0$, giving $\Pi_1(\phi,r) = \alpha_1(r)$.

Now $\alpha_1(r)\, r^{-(\sigma+1)}$ can be incorporated in $\Pi_0(r)$ and then the next term in the expansion for p is $\dfrac{\Pi_2(\phi,r)}{r^{2(\sigma+1)}}$ where

$$\frac{\partial \Pi_2(\phi,r)}{\partial \phi} = \frac{2u_1(\phi,r)}{(\gamma+1)} \quad .$$

Similarly the term $R_1(\phi,r)\, r^{-(\sigma+1)}$ can be absorbed by $R_0(\phi,r)$ and the next term is then $\dfrac{R_2(\phi,r)}{r^{2(\sigma+1)}}$ where

$$R_2(\phi,r) = \frac{R_0(\phi,r)\,\Pi_2(\phi,r)}{\gamma\, \Pi_0(r)} \quad .$$

The functions $f_i(r)$ will now include $\alpha_1(r)r^{-(\sigma+1)}$ and they will be restricted by

$$O(1) > f_i(r) > O(r^{-2(\sigma+1)}) \quad \text{for all } i \text{ as } r \to \infty.$$

We conclude this section by saying that the boundary condition at the contact front gives

$$u_1(1,r) = 0 \,, \quad k \neq (\sigma+1) \,,$$

$$u_1(0,r) = 0 \,, \quad k = (\sigma+1) \,.$$

5.2. The breakdown of the outer solution as the shock is approached.

For large r we let

$$V(r) = b_0 \left\{ 1 + \frac{b_1}{r^w} + \ldots\ldots \right\} \,, \qquad (5.2.1)$$

b_0 , b_1 , $w > 0$ being unknown constants. Then after manipulation we can obtain leading terms in series solutions for the gas variables as we approach the inner limit of the outer expansions. These series are shown in the appendix, AI, and we demonstrate here with u_1 . For $0 \leq k < (\sigma+1)$, as $\phi \to \infty$,

$$u_1(\phi,r) = O(\phi^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}}) \,, \qquad (5.2.2)$$

for $k = (\sigma+1)$, as $\psi \to \infty$,

$$u_1(\psi,r) = O(e^{k(\gamma-1)\psi/\gamma}) \,, \qquad (5.2.3)$$

for $k > (\sigma+1)$, as $\phi \to 0$, we have (5.2.2) again.

It can now clearly be seen that these outer expansions will no longer be valid when

$$\phi = O(r^{\sigma+1-k}) \quad , \quad k \neq (\sigma+1) \quad ,$$

$$\psi = O(\ln r) \quad , \quad k = (\sigma+1) \quad .$$

We therefore replace the outer variable, ϕ or ψ, in these asymptotic expansions by the inner variable, ϕ_1 , defined by equations (4.3.1). In this way we can now produce the inner limit of the outer expansions. These are

$$\frac{u}{b_0} = U_0(\phi_1) + U_0^{*}(\phi_1, r) + \frac{U_1(\phi_1)}{r^{\sigma+1-k/\gamma}} + \frac{U_1^{*}(\phi_1, r)}{r^{\sigma+1-k/\gamma}} + \frac{b_1 U_2(\phi_1)}{r^{w}} + \ldots\ldots \quad , \qquad (5.2.4)$$

for all k, with similar expressions for

$\frac{2}{\gamma+1} b_0^{-2} r^k p$ and $\left(\frac{\gamma-1}{\gamma+1}\right) r^k \rho$. The starred quantities are $o(1)$ whereas the others are $O(1)$.

Series expansions for the above quantities, as ϕ_1 approaches its outer limit, are presented in the appendix, AII . These expressions together with the series expansions give the matching conditions for the inner expansions as

$$\phi_1 \to 0 \quad , \quad k \leq (\sigma+1) \quad ,$$

$$\phi_1 \to \infty \quad , \quad k > (\sigma+1) \quad .$$

5.3. The inner solution and matching of the two expansions.

In this section we produce asymptotic expansions for solutions in the inner region and then we proceed to match them with the outer expansions. Matching to zeroth order provides a means of calculating the asymptotic shock velocity while matching to first order presents an eigenvalue problem.

The inner equations are (4.3.2), (4.3.3) and the boundary conditions at the shock, $\phi_1 = 1$, are presented in (4.2.7). We seek asymptotic solutions of these inner equations, for $\phi_1 = O(1)$, of the form

$$\left.\begin{aligned}
V_1(r) &= \frac{(\gamma+1)}{2}\, b_0 \left\{1 + \frac{b_1}{r^w} + \ldots\ldots\right\} , \\
u(\phi_1, r) &= b_0 \left\{U_0(\phi_1) + \frac{b_1\ U_i(\phi_1)}{r^{\alpha}} + \ldots\ldots\right\} , \\
p(\phi_1, r) &= \frac{(\gamma+1)}{2}\, b_0^2\, r^{-k} \left\{ P_0(\phi_1) \right. \\
&\qquad \left. + \frac{b_1\ P_i(\phi_1)}{r^{\alpha}} + \ldots\ldots\right\} , \\
\rho(\phi_1, r) &= \frac{(\gamma+1)}{(\gamma-1)}\, r^{-k} \left\{ \rho_0(\phi_1) \right. \\
&\qquad \left. + \frac{b_1\ \rho_i(\phi_1)}{r^{\alpha}} + \ldots\ldots\right\} .
\end{aligned}\right\} \quad (5.3.1)$$

Inserting these expansions into the inner equations and equating coefficients of appropriate powers of r to zero gives

$$\phi_1 \left[U_0\rho_0' + \rho_0 U_0'\right] - \frac{(\gamma+1)}{(\gamma-1)} \rho_0^2\, \beta(\phi_1) U_0' - \frac{(\sigma-k)}{\mu} \rho_0 U_0 = 0,$$

$$\left.\begin{aligned}
&\rho_0 U_0 \phi_1 U_0' + \frac{k(\gamma-1)}{2\mu} P_0 + \frac{(\gamma-1)}{2}\phi_1 P_0' \\
&\qquad\qquad - \frac{(\gamma+1)}{2}\rho_0 \beta(\phi_1) P_0' = 0, \\
&P_0 = \rho_0^{\gamma}\, \phi_1^{\frac{k(\gamma-1)}{\mu}} ,
\end{aligned}\right\} \quad (5.3.2)$$

$$\left.\begin{aligned}
&\phi_1 \left[U_0 \rho_i' + U_i \rho_0' + \rho_0 U_i' + \rho_i U_0'\right] \\
&- \frac{(\gamma+1)}{(\gamma-1)} \beta(\phi_1)\left[\rho_0^2\, U_i' + 2\rho_0 \rho_i U_0'\right] \\
&- \frac{(\sigma-k-\alpha)}{\mu}\left[\rho_0 U_i + \rho_i U_0\right] = 0, \\
&\phi_1 \left[\rho_0 U_0 U_i' + \rho_0 U_i U_0' + \rho_i U_0 U_0'\right] + \frac{\alpha}{\mu}\rho_0 U_0 U_i \\
&+ \frac{(k+\alpha)(\gamma-1)}{2\mu} P_i + \frac{(\gamma-1)}{2}\phi_1 P_i' \\
&- \frac{(\gamma+1)}{2}\beta(\phi_1)\left[\rho_0 P_i' + \rho_i P_0'\right] = 0, \\
&P_i = P_0\left[\frac{\gamma \rho_i}{\rho_0} + 2\,\phi_1^{-w/\mu}\, \delta_w^{\alpha}\right],
\end{aligned}\right\} \quad (5.3.3)$$

where we define

$$\beta(\phi_1) = \begin{cases} 1 , & k \neq (\sigma+1) , \\ \phi_1 , & k = (\sigma+1) , \end{cases}$$

$$\left.\mu = \begin{cases} \sigma+1-k , & k \neq (\sigma+1) , \\ 1 , & k = (\sigma+1) , \end{cases}\right\} \quad (5.3.4)$$

$$\delta_w^{\alpha} = \begin{cases} 0 \ , & \text{if } \alpha < w \ , \ i = 1 \ , \\ 1 \ , & \text{if } \alpha = w \ , \ i = 2 \ . \end{cases}$$

The error terms of order $r^{-\alpha}$, $\alpha < w$, typify some of the indeterminate terms carried over from the outer solution.

The boundary conditions, (4.2.7), give

$$U_0(1) = P_0(1) = \rho_0(1) = 1 \quad , \qquad (5.3.5)$$

$$U_1(1) = \delta_w^{\alpha} \ , \ P_1(1) = 2\delta_w^{\alpha} \ , \ \rho_1(1) = 0 \quad , \qquad (5.3.6)$$

It can be seen, on examining the inner limit of the outer expansions given in the appendix, that we have omitted the particular case of $w = \sigma+1-k/\gamma$. If we had allowed for this then we would have a term of order $\dfrac{b_1 \ln r}{r^{\sigma+1-k/\gamma}}$ in $U_0^*(\phi_1, r)$ which cannot be removed by any other terms there nor, as we show later, has it any matching term in the inner solution. If we had allowed for an error term in the shock velocity of the form $\dfrac{b_1 (\ln r)^m}{r^{\sigma+1-k/\gamma}}$ the result would be an irremoveable term in $U_0^*(\phi_1, r)$ of order $\dfrac{b_1 (\ln r)^{m+1}}{r^{\sigma+1-k/\gamma}}$ having no matching term in the inner solution. This feedback of logarithmic dependence removes the possibility of the first order error term in the shock velocity being of order $r^{-(\sigma+1-k/\gamma)}$. It is for similar reasons that we also ignore the case $w = 2(\sigma+1) - k(\gamma+1)/\gamma$, for $k > (\sigma+1)$, since this would introduce irremoveable terms in $P_0^*(\phi_1, r)$ which have no matching components in the inner solution.

We have said that there are no matching terms for the logarithmic behaviour. This is a consequence of the solution of the first order inner problem being identically zero for $\alpha < w$, $i = 1$. For this case it is clear that, since the equations are linear and homogeneous in $U_1(\phi_1)$, $P_1(\phi_1)$, $\rho_1(\phi_1)$ and the boundary conditions present zero values for these three quantities at $\phi_1 = 1$, the solutions for $U_1(\phi_1)$, $P_1(\phi_1)$, $\rho_1(\phi_1)$ are identically zero.

Now if $w > (\sigma+1-k/\gamma)$ then we can see that, since there will be no terms in the inner solution between $O(1)$ and $O(r^{-w})$. $U_0^* = U_1 = U_1^* = 0$, etc.. This indicates that, in AI and AII, $G_0(r) = 0$ and hence $F_0(r) = 1$ with G_1, G_3, H_0, H_1, H_2, $\alpha_1(r)$, H_3, G_2 all zero. The solutions $U_1 = P_1 = P_1 = 0$ give us conditions on certain integrals involving the shock velocity which must be met for a successful match of the two sets of expansions up to this order.

These conditions can be written in the form

$$\int_1^{\infty} r^{\sigma-k/\gamma} \left\{ \left[\frac{V(r)}{b_0}\right]^{\frac{2}{\gamma}} - 1 \right\} dr = \frac{1}{\sigma+1-k/\gamma} ,$$

for all k , together with

$$\int_1^{\infty} r^{k(\gamma-1)/\gamma} \ln r \left\{ \left[\frac{V(r)}{b_0}\right]^{\frac{2}{\gamma}} - 1 \right\} \frac{dr}{r} = \frac{-\gamma^2}{k^2(\gamma-1)^2} ,$$

for $k = (\sigma+1)$, and

$$\int_1^{\infty} r^{2(\sigma+1)-k(\gamma+1)/\gamma} \left\{ \left[\frac{V(r)}{b_0}\right]^{\frac{2}{\gamma}} - 1 \right\} \frac{dr}{r} = \frac{1}{2(\sigma+1) - k(\gamma+1)/\gamma} ,$$

for $k > (\sigma+1)$.

Looking for series solutions of the zeroth order inner problem valid near the outer limit of ϕ_1 , we find, from (5.3.2),

$$U_0(\phi_1) = \frac{1}{b_0}\left\{1 - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{\sigma+1-k/\gamma}{\mu}} + \ldots\right\} ,$$

$$P_0(\phi_1) = \left(\frac{\alpha_0}{b_0^2}\right)\left\{1 + \frac{p_0}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\nu} + \ldots\ldots\right\} ,$$

and then $\rho_0(\phi_1)$ given from the third of (5.3.2), where μ is defined in (5.3.4),

$$p_0 = \begin{cases} \dfrac{k(\gamma-1)}{(\sigma+1-k/\gamma)} , & k < (\sigma+1) , \\ \gamma\left[1 - \dfrac{2(\sigma-k/\gamma)}{k\alpha_0(\gamma+1)}\right] , & k = (\sigma+1) , \\ -\dfrac{2(\sigma-k/\gamma)(\gamma-1)}{\alpha_0(\gamma+1)\left[2(\sigma+1)-k(\gamma+1)/\gamma\right]} , & k > (\sigma+1) , \end{cases}$$

and

$$\nu = \begin{cases} \dfrac{\sigma+1-k/\gamma}{\mu} , & k \lesssim (\sigma+1) , \\ \dfrac{2(\sigma+1)-k(\gamma+1)/\gamma}{\sigma+1-k} , & k > (\sigma+1) . \end{cases}$$

It can clearly be seen that these terms match with terms in the outer solution since we have anticipated the matching condition, b_0^{-1} and $\alpha_0 b_0^{-2}$, on U_0 and P_0 respectively in the outer limit.

The solutions to the zeroth order problem are the similarity solutions of section 3.3.. However the Lagrangian description was chosen for numerical integration in preference to the similarity solution since, in the latter for $k > 0$, the contact front presents a singularity in the differential equations describing the solution.

In order to determine b_0 we define a new function, related to ρ_0, and choose a new independent variable to overcome the problems involved when derivatives become singular at the outer limit of ϕ_1. We let

$$\rho_0 = S_0\, \phi_1^{\frac{-k(\gamma-1)}{\gamma\mu}} ,$$

$$x = \begin{cases} \phi_1^{\frac{\sigma+1-k/\gamma}{\mu}} , & k \leq (\sigma+1) , \\ \phi_1^{\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{\sigma+1-k}} , & k > (\sigma+1) . \end{cases}$$

The third equation of (5.3.2) now becomes $P_0 = S_0^{\gamma}$ and, on using this, the first two equations of (5.3.2) can be rearranged as

$$\left.\begin{aligned} \frac{dU_0}{dx} &= \frac{U_0 S_0^{\gamma}\, x^{\alpha_1}\left\{ kx^{\alpha_2} - \gamma(\sigma-k/\gamma)\, T\right\}}{D\left\{\gamma S_0^{\gamma}\, T^2 - \frac{2}{(\gamma-1)}\, S_0 U_0^2\, x^{\alpha_3}\right\}} , \\ \frac{dS_0}{dx} &= \frac{S_0\left\{ kS_0^{\gamma}\, T\, x^{\alpha_1} - \frac{2}{(\gamma-1)}(\sigma-k/\gamma)\, S_0 U_0^2\, x^{\alpha_3-1}\right\}}{D\left\{\gamma S_0^{\gamma}\, T^2 - \frac{2}{(\gamma-1)}\, S_0 U_0^2\, x^{\alpha_3}\right\}} , \end{aligned}\right\} \quad (5.3.7)$$

where $$\alpha_1 = \begin{cases} 0 & , k \leq (\sigma+1) \quad , \\ \dfrac{(k-\sigma-1)}{2(\sigma+1)-k(\gamma+1)/\gamma} & , k > (\sigma+1) \quad , \end{cases}$$

$$\alpha_2 = \alpha_1 + 1 \quad ,$$

$$\alpha_3 = \left\{\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{(\sigma+1-k/\gamma)}\right\}\alpha_2 \quad ,$$

$$D = \begin{cases} \sigma+1-k/\gamma & , k \leq (\sigma+1) \quad , \\ 2(\sigma+1)-k(\gamma+1)/\gamma & , k > (\sigma+1) \quad , \end{cases}$$

$$T = \frac{(\gamma+1)}{(\gamma-1)} S_0 - x^{\alpha_2} \quad .$$

The boundary conditions are $U_0 = S_0 = 1$ on $x = 1$, while matching requires $U_0 = b_0^{-1}$ at $x = 0$.

Whatever value is assigned to k, it is a simple exercise to solve for b_0 by numerically integrating equations (5.3.7) simultaneously from $x = 1$ to $x = 0$ thus giving $b_0 = \dfrac{1}{U_0(0)}$ and also $\alpha_0 = b_0^2 S_0(0)$, where U_0 and S_0 are now functions of x. Typical graphs of a_0 as a function of k for various values of γ, σ are shown in FIGS. 14 to 16, where $a_0 = \frac{(\gamma+1)}{2} b_0$.

We now turn our attention to the first non-zero error terms in the inner solution and proceed to match it with terms in the outer solution. The relevant differential equations, (5.3.3) with $\alpha = w$ and $i = 2$, are linear and inhomogeneous in U_2 , P_2 , ρ_2 and we can therefore split up the solution into two parts: the particular integral corresponding to the inhomogeneous term in (5.3.3) and the complementary function, the general solution of the homogeneous equations. The arbitrary constants which arise in the complementary function must be chosen so that the boundary conditions at the shock are satisfied.

The leading terms in series expansions of the particular integral parts of U_2 and P_2 are, in the outer limit of ϕ_1 ,

$$U_2^{(1)}(\phi_1) = \frac{2(\sigma-k/\gamma)(\gamma-1)}{\gamma(\gamma+1)\left[w-(\sigma+1-k/\gamma)\right] b_0} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{-w+(\sigma+1-k/\gamma)}{\mu}} + \ldots ,$$

$$P_2^{(1)}(\phi_1) = \frac{-2\, p_2\, (\gamma-1)}{\gamma(\gamma+1)\left[w-(\sigma+1-k/\gamma)\right]} \phi_1^{\alpha_4} + \ldots ,$$

where
$$p_2 = \begin{cases} k\left(\dfrac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} , & k < (\sigma+1) , \\[2ex] \dfrac{k\left\{2\left(\frac{\gamma-1}{\gamma+1}\right)(\sigma-k/\gamma) + \alpha_0\left[w-k(\gamma-1)/\gamma\right]\right\}}{\left[w-k(\gamma-1)/\gamma\right] b_0^2} , & k = (\sigma+1) , \\[2ex] \dfrac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)}{(\gamma+1)\left[w-2(\sigma+1)+k(\gamma+1)/\gamma\right] b_0} \left(\dfrac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} , & k > (\sigma+1) , \end{cases}$$

$$\alpha_4 = \begin{cases} \dfrac{-w+(\sigma+1-k/\gamma)}{\mu} , & k \lesssim (\sigma+1) , \\[2ex] \dfrac{w-2(\sigma+1)+k(\gamma+1)/\gamma}{(k-\sigma-1)} , & k > (\sigma+1) . \end{cases}$$

The expansions for $\rho_2^{(1)}(\phi_1)$ can be found by using that of $P_2^{(1)}(\phi_1)$ in the third of equations (5.3.3).

As expected the complementary function involves two arbitrary constants and the leading terms of this part of the solution for U_2 , P_2 are, in the outer limit of ϕ_1 , .

$$U_2^{(2)}(\phi_1) = \frac{A_4}{b_0} + \ldots\ldots \quad , \text{ for all } k,$$

$$P_2^{(2)}(\phi_1) = \begin{cases} \gamma \left(\dfrac{\alpha_0}{b_0^2}\right) B_4 + \ldots\ldots & , \text{ for } k \lesssim (\sigma+1) , \\[2ex] \dfrac{-2w\ A_4\ \phi_1}{(\gamma+1)(k-\sigma-1)\ b_0^2} + \gamma\left(\dfrac{\alpha_0}{b_0^2}\right)B_4 + \ldots & , \text{ for } k > (\sigma+1), \end{cases}$$

Again the corresponding form for $\rho_2^{(2)}(\phi_1)$ can easily be obtained. The constants A_4 and B_4 are independent and arbitrary.

The full expressions for U_2 , P_2 , ρ_2 are

$$U_2\ (\phi_1) = U_2^{(1)}(\phi_1) + U_2^{(2)}(\phi_1) \quad ,$$

$$P_2\ (\phi_1) = P_2^{(1)}(\phi_1) + P_2^{(2)}(\phi_1) \quad ,$$

$$\rho_2\ (\phi_1) = \rho_2^{(1)}(\phi_1) + \rho_2^{(2)}(\phi_1) \quad .$$

The terms from the particular integral automatically match with terms in the outer solution for any value of w but the matching of terms from the complementary function is not as straightforward. For all k the leading term in $U_2^{(2)}(\phi_1)$ is a constant. If we now express the inner expansion of u in terms of the outer variables we obtain

$$u = 1 + \frac{A_4}{r^w} + \frac{u_1}{r^{\sigma+1-k/\gamma}} + \frac{u_2}{r^{2(\sigma+1-k/\gamma)}} + \ldots\ldots \quad ,$$

where u_1 and u_2 are functions of the outer variables and are $O(1)$ in the outer region. Now there can be no terms between $O(1)$ and $O(r^{-(\sigma+1-k/\gamma)})$ for any w and we have already shown that, if $w > (\sigma+1-k/\gamma)$, u_1 is independent of r and there can be no terms between $O(r^{-(\sigma+1-k/\gamma)})$ and $O(r^{-2(\sigma+1-k/\gamma)})$. We conclude, therefore, that, if $w < 2(\sigma+1-k/\gamma)$, $A_4 = 0$. The implication of this is that b_1 is indeterminate and that w is an eigenvalue which takes a value, complex in general, such that the finite part of $U_2(\phi_1)$, that is $\frac{A_4}{b_0}$ is identically zero. If w is complex then so are U_2 , P_2 , ρ_2 and we take the perturbation to the zeroth order inner solution to be the real parts of $U_2 r^{-w}$ etc.. An obvious requirement of w is that its real part must be negative and the most interesting of these eigenvalues is the one with the largest real part. The solution for w is rather complicated but we give an outline of a possible method for its evaluation. This method is the one used by Stewartson & Thompson (1970) who sought perturbations to the blast wave solution in the unsteady analogue of hypersonic flow past blunt bodies. The computational work is difficult because the singular parts of U_2 , P_2 , ρ_2 must be removed so that the integration can be carried down to the neighbourhood of the contact front. This can only be done in an approximate fashion since rounding off must take place and any numerical integration technique will have inherent truncation errors. Consequently as we approach the outer limit of ϕ_1 , these errors will become significant for they can be associated with unremoved singularities with very small coefficients. Thus, however small these coefficients are, they will dominate the solution near the contact front and introduce instability to the numerical integration and also the iterative method involved in determining the value of $\dot{w}$. To overcome this Stewartson & Thompson integrated their equations and took the finite part to be the last value before instability set in.

Returning to the zeroth order inner problem, when solving equations (5.3.7) numerically it was found that there was an upper limit, k_c , to k for a successful integration. This was indicated by instability in the solution as ϕ_1 approached its outer limit but to define k_c clearly and also to evaluate it we draw on information derived in section 3.3. for the similarity solutions.

5.4. Determination of k_c from the similarity solutions.

As stated there is an upper limit, k_c , to k and we determine k_c , as a function of γ and σ, in this section by inspecting the integral curves of section 3.3..

We extended the work of Sedov and introduced Γ, the integral curve connecting points C and D in the phase plane of Z and V.

If $k \geqslant (\sigma+1)$ and $0 < \sigma \leqslant 1$ then $V_B < V_S$ and $Z_B > Z_S$ while for $k \geqslant (\sigma+1)$ and $\sigma > 1$ then $Z_B < 0$. Thus for $k \geqslant (\sigma+1)$ and $\sigma > 0$ singular point B never lies in the range $V_S \leqslant V \leqslant 1$.

Now for $k = (\sigma+1)$ and $\sigma > 0$, Γ lies to the left of S and we can construct a solution curve between S and C. As k increases the curve, Γ , moves to the right until, at $k = k_c$, it passes through S. For $k > k_c$ there will be no integral curve connecting S to C. Thus we have a clear indication of k_c and further we note that B and S coincide for $\sigma = 0$, $k = 1$, for all γ and therefore $k_c = 1$ for $\sigma = 0$, any γ.

The evaluation of k_c must be done numerically but we can produce bounds on it. We shall assume at this stage that, for $k = k_c$, $V_A > V_S$ and so $\max(V_B, V_D) < V_S < \min(1, V_A)$. Now we have shown in FIGS. 3 to 6 that, for any V in this range, Γ lies above the locus of infinite slope, $Z = Z_\infty(V)$, and thus $Z_S > Z_\infty(V_S)$. This inequality, together with $k_c \geq (\sigma+1)$, gives

$$(\sigma+1) \leq k_c \leq \frac{2\gamma\sigma}{\gamma+1} + 1 \;, \quad \text{the equality only holding for}$$

$\sigma = 0$.

These somewhat crude bounds can be refined by some iterative scheme and the one chosen was the bisection method merely because of its simplicity and reliability. Suppose, at some stage, that the lower and upper bounds on k_c are k_0 and k_1 respectively. Let $k_2 = \frac{1}{2}(k_0+k_1)$ and construct Γ, using $k = k_2$, from A, if $V_A < 1$, or C otherwise up to $Z = Z_s$ giving $V = V_1$ at this point on the curve. It is plain to see that if $V_1 < V_S$ then $k_2 < k_c$ and we replace k_0 by k_2 while if $V_1 > V_S$ then $k_2 > k_c$ and we replace k_1 by k_2 . This algorithm is repeated until we have sufficiently fine bounds on k_c.

The curve, Γ, is constructed by numerically integrating equation (3.3.1) if $V_A < 1$ or the equation

$$\frac{dY}{dV} = \frac{1}{2Y}\frac{dZ}{dV} \;, \text{ where } Z = Y^2 \text{ and } \frac{dZ}{dV} \text{ is given by (3.3.1).}$$

The integration technique used was the fourth order Runge-Kutta method with Gill's modification.

The curve $k = k_c(\gamma, \sigma)$ in the $k-\sigma$ parameter space is shown in FIGS. 17,18 for $\gamma = \frac{7}{5}$, $\frac{5}{3}$ respectively and it shows, for the range of σ taken, that $k_c > \frac{2\gamma\sigma}{\gamma+1}$, that is $V_A > V_S$, agreeing with the assumption made earlier in the production of an initial bound on k_c.

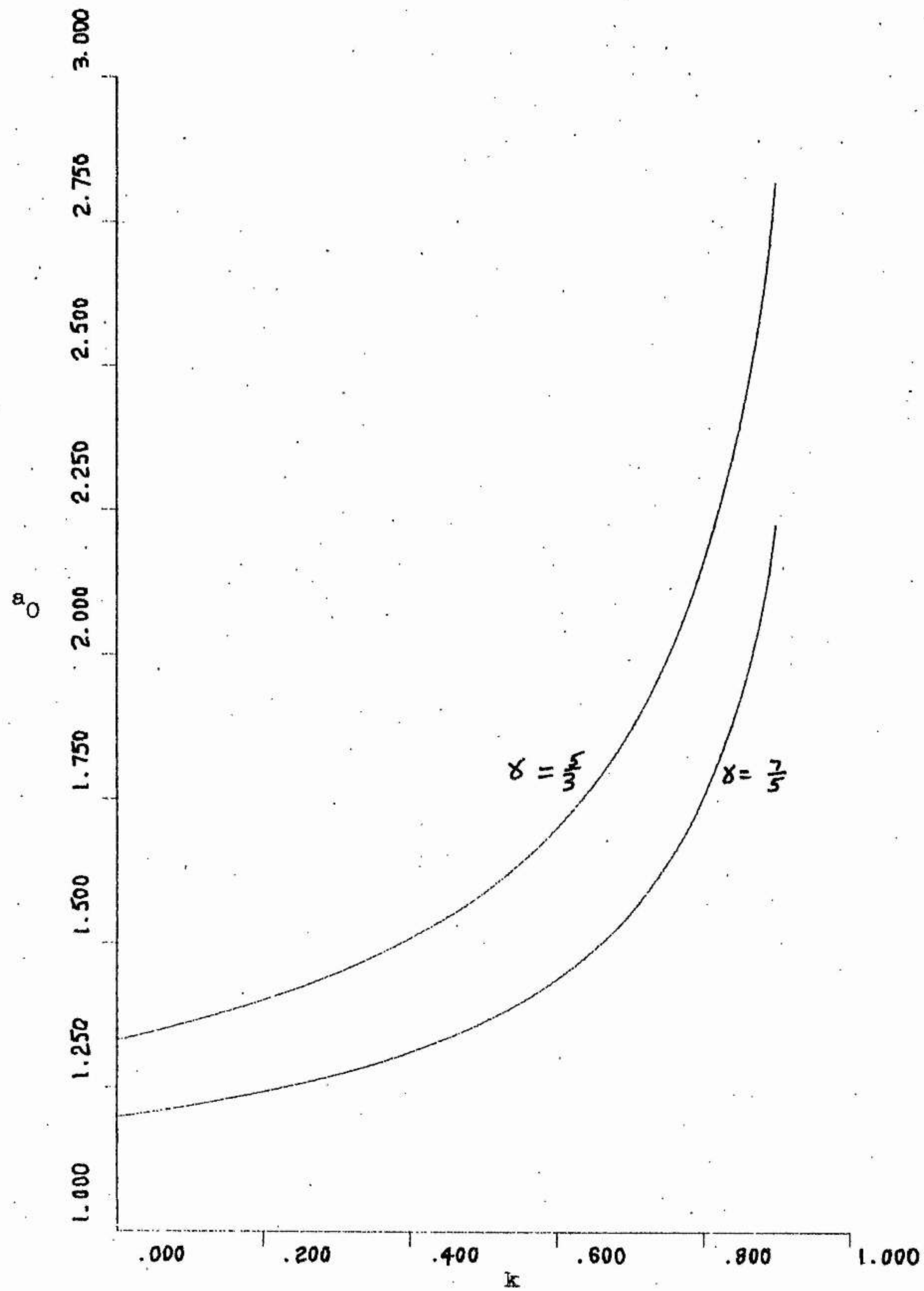


FIG. 14. $\sigma = 0$.

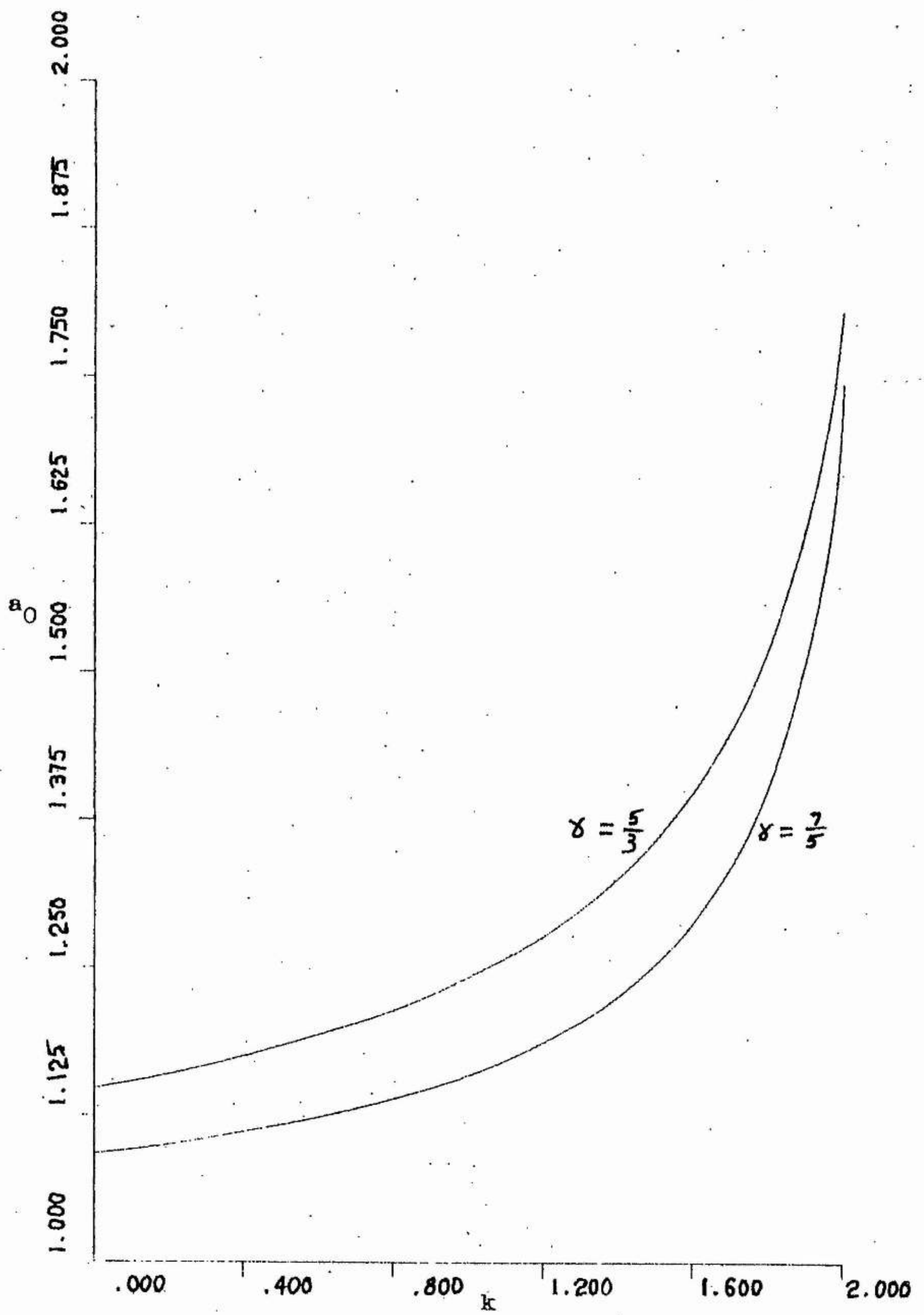


FIG. 15. $\sigma = 1$.

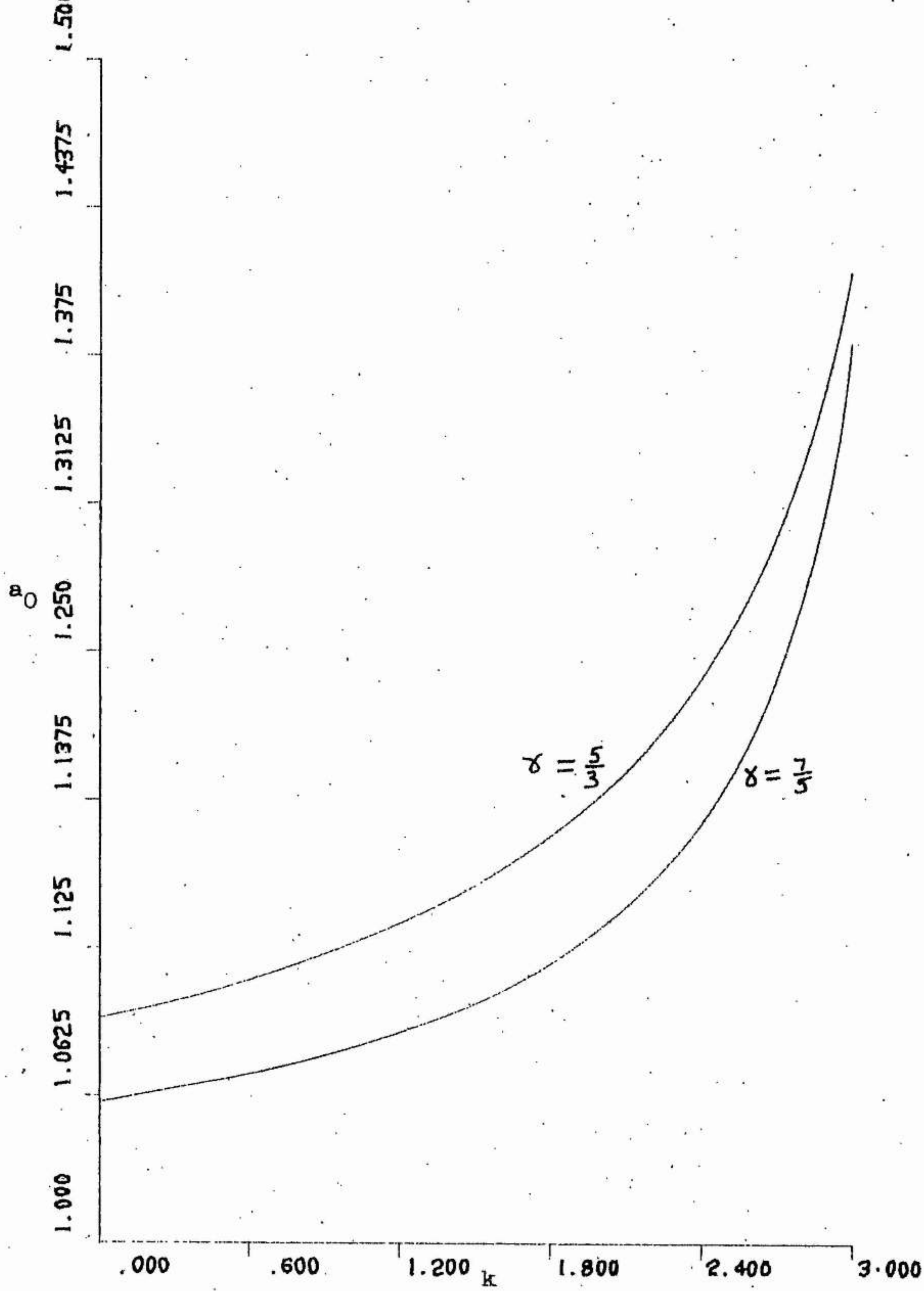


FIG. 16. $\sigma = 2$.

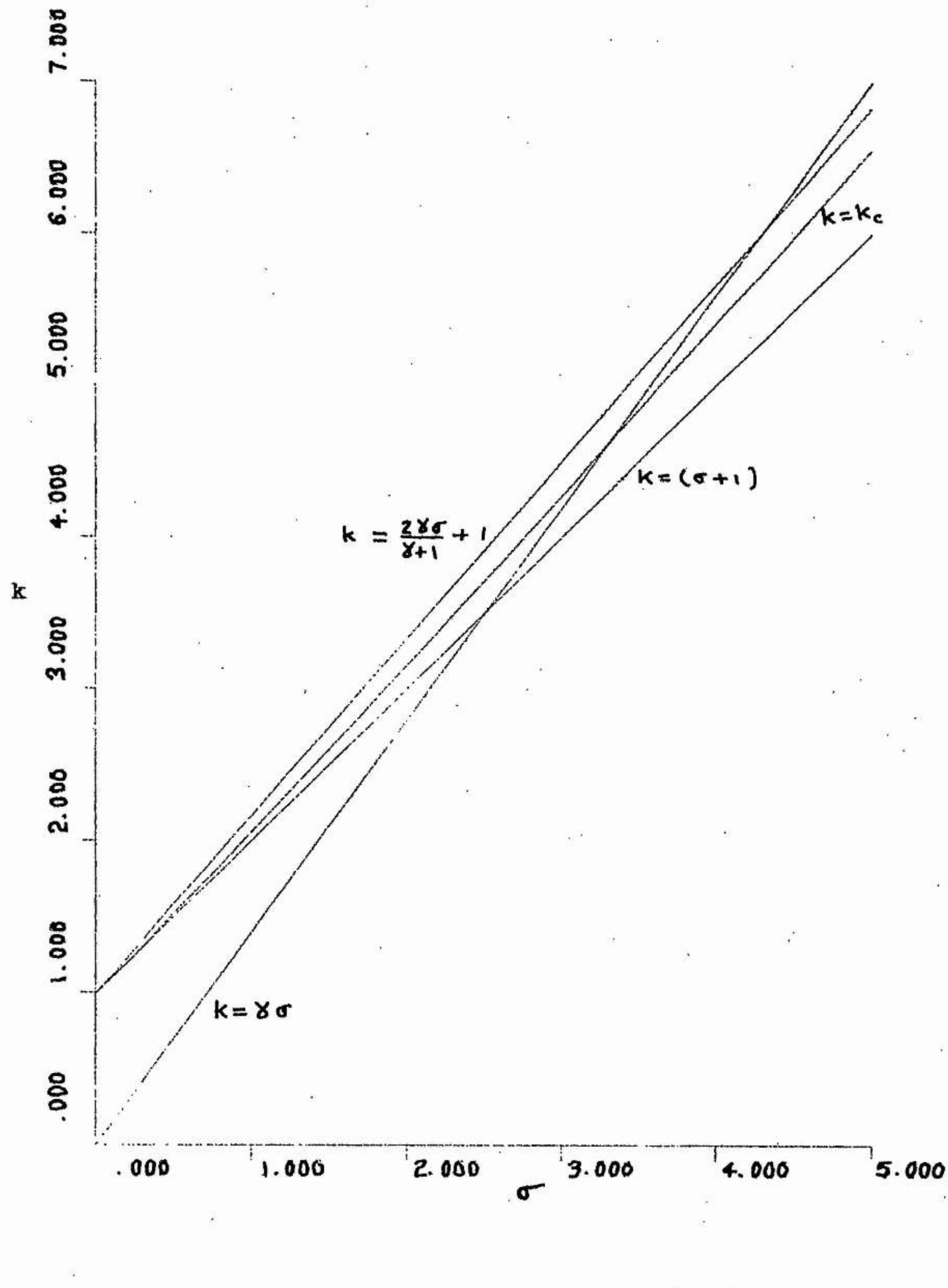


FIG. 17. k−σ parameter space for $\gamma=7/5$.

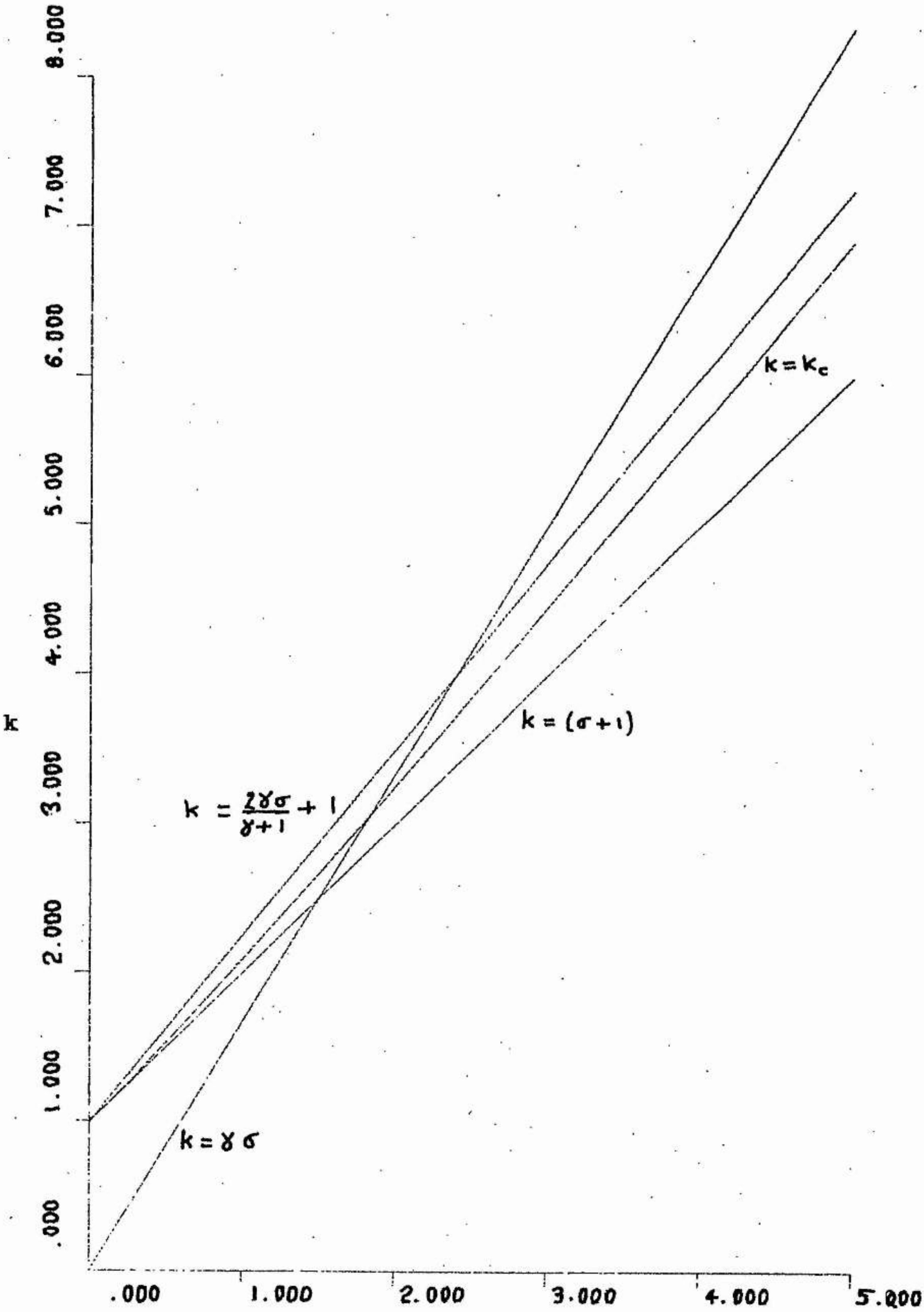


FIG. 18. k-σ parameter space for $\gamma = 5/3$.

6. Matched asymptotic expansions for $k > k_c$.

6.1. Investigation of possible asymptotic solutions and the evaluation of δ.

In this section we look at possible forms for an asymptotic solution when $k > k_c$. The search for an acceptable asymptotic solution involved some blind alleys and it is relevant, and perhaps necessary, to list some of these since the search serves as a process of elimination and it shows how we arrived at an understanding of the problem.

In chapter 5 we showed that $k = k_c$ is the limit for an inner solution of the type shown there to exist. However no such restriction was placed on the outer solution. It may seem, then, that we can produce outer expansions which are essentially those of (5.1.10) . If we use (5.1.10) as a starting point, together with $V(r) = b_0\, r^{\epsilon}\left\{1 + \dfrac{b_1}{r^w} + \ldots\ldots\right\}$, as $r \to \infty$, we find that these outer expansions breakdown when

$$\phi = O(r^a) \;, \quad a = \frac{-(k-\sigma-1)\,\left[2\gamma(\sigma+1)-k(\gamma+1)\right]}{\left[2\gamma(\sigma+1)+2\epsilon-k(\gamma+1)\right]} \quad .$$

This breakdown cleary occurs before the inner region is reached since $\epsilon = 1-\delta^{-1} > 0$ and this means that another layer, which we shall call Intermediate I, must be inserted. In this layer the appropriate independent variables are ϕ_2, r where $\phi_2 = \phi r^{-a}$.

Using matching conditions from the outer region we can construct asymptotic expansions in this region and it can be shown that they breakdown when

$$\phi = O(r^b) \quad , \quad b = \frac{-(k-\sigma-1)(\gamma-1)(\sigma+1)}{\left[2\epsilon+(\gamma-1)(\sigma+1)\right]} \quad .$$

Again this breakdown occurs before the inner region is reached and another layer, Intermediate II, must be included in the flow field. In this region the appropriate variables are ϕ_3 and r where $\phi_3 = \phi r^{-b}$.

Writing the expansions valid in the Intermediate I region in terms of the Intermediate II variables produces matching conditions.

In the Intermediate II region we expand, for $\phi_3 = O(1)$, $r \to \infty$,

$$u = u_0^{*}(\phi_3) + \ldots\ldots \quad ,$$

$$p = \frac{(\gamma+1)}{2} r^{-\alpha} \Pi_0^{*}(\phi_3) + \ldots\ldots \quad ,$$

$$\rho = \frac{(\gamma+1)}{(\gamma-1)} r^{-\alpha} R_0^{*}(\phi_3) + \ldots\ldots \quad ,$$

where $\alpha = -k + \dfrac{2\epsilon(k-\sigma-1)}{2\epsilon + (\gamma-1)(\sigma+1)}$,

and then seek series expansions of these zeroth order terms as $\phi_3 \to 0$. This gives

$$u_0^{*}(\phi_3) = A_0^{*} \phi_3^{-c} + \ldots\ldots \quad ,$$

$$\Pi_0^{*}(\phi_3) = b_0^2 \, B_0^{*\gamma} \phi_3^{-d} + \ldots\ldots \quad ,$$

$$R_0(\phi_3) = B_0^{*} \phi_3 + \ldots\ldots \quad ,$$

where $c = \dfrac{(\gamma-1)(\sigma+1)+2}{2(k-\sigma-1)}$,

$$\alpha = \frac{\gamma(\sigma+1)+2\epsilon-k}{(k-\sigma-1)} \quad ,$$

$$B_0^{*} = \frac{(\gamma-1)\ \left[2+(\gamma-1)(\sigma+1)\right]}{(\gamma+1)\ \left[2\epsilon+(\gamma-1)(\sigma+1)\right]} \quad ,$$

$$A_0^{*2} = \frac{b_0^2\ B_0^{*\gamma-1}\ \left\{2\gamma(\sigma+1)+2\epsilon(1-\sigma)-k\left[2+(\gamma-1)(\sigma+1)\right]\right\}}{(\sigma+1)\ \left[2\epsilon+(\gamma-1)(\sigma+1)\right]} \quad .$$

Now we assume that the Intermediate II region overlaps the inner region where the inner variables ϕ_1 and r are relevant.

Expressing the Intermediate II solutions in terms of the inner variables produces matching conditions

$$u = r^{\epsilon}\left\{A_0^{*}\phi_1^{-c} + \ldots\ldots\right\} \quad ,$$

$$p = \frac{(\gamma+1)}{2}\ b_0^2\ r^{2\epsilon-k}\left\{B_0^{*\gamma}\phi_1^{-\alpha} + \ldots\ldots\right\} \quad , \qquad (6.1.1)$$

$$= \frac{(\gamma+1)}{(\gamma-1)}\ r^{-k}\left\{B_0^{*}\phi_1 + \ldots\ldots\right\} \quad .$$

It can be shown that these expressions satisfy the zeroth order inner equations for $\phi_1 \to \infty$.

This analysis appears, at first sight, to produce an asymptotic solution to the problem but further examination proves otherwise. In particular the coefficient A_0^{*} becomes imaginary for

$$k > \frac{2\gamma(\sigma+1) + 2\epsilon(1-\sigma)}{2 + (\gamma-1)(\sigma+1)} \quad ,$$

assuming that $b_0\ B_0^{*\left(\frac{\gamma-1}{2}\right)}$ is real, and this is precisely the condition for the Z coordinate of singular point B in the phase plane of the zeroth order inner problem to be negative. An examination of equation (4.3.4) in this instance shows the following, for $\phi_1 \to \infty$,

$$\delta \lambda^{\frac{1}{\delta}} = \frac{\frac{(\gamma+1)}{(\gamma-1)} B_0^{*} b_0 U_0}{\frac{(\gamma+1)}{(\gamma-1)} B_0^{*} - 1} + \ldots\ldots$$

$$= \tfrac{1}{2}\delta\left[2 + (\gamma-1)(\sigma+1)\right] b_0 U_0 + \ldots\ldots \quad .$$

Now $b_0 U_0 = \delta\lambda^{\frac{1}{\delta}} V$, as $r \to \infty$, and thus we have

$$V = \frac{2}{\delta\left[2 + (\gamma-1)(\sigma+1)\right]} + \ldots\ldots$$

$$= V_B + \ldots\ldots \quad .$$

It appears, therefore, that we are attempting to match at singular point B. An inspection of the integral curves in the phase plane of Z and V shows that this is physically impossible since $\lambda \to \infty$ at B if $Z_B > 0$ but, since the contact front is behind the shock, we must leave S, in the phase plane, along an integral curve such that λ is decreasing. The consequence of this is that there is also the same limit, $k = k_c$, for the outer solution of section 5.1., and therefore the solution near the contact front is different for $k > k_c$ to that for $k < k_c$.

Since we draw no definite conclusions on the outer solution at this stage it seems reasonable to concentrate first of all on the inner region near the shock. With λ decreasing away from S along the solution curve we can see that matching can only take place at one of the singular points C, G ,F.

An investigation of the solution with matching at singular point C produces a velocity in the outer region near the contact front which is $O(r^{\epsilon})$. If we now try to enforce the boundary condition, $u = 1$, on the contact front we have an inconsistency since it gives $b_0 = 0$. This will, however, furnish a solution to the case of a piston moving outwards with a speed proportional to r^{ϵ}.

We noted, in section 3.4., that there is an integral curve, named Δ, emanating from F passing through G with λ increasing monotonically. It appears, therefore, that F is the point at which to perform the matching provided that we satisfy a certain condition on δ. The condition is that we adjust δ so that the curve Δ also passes through S, the position of the strong shock. This now gives us a means of determining δ as a function of k, γ, σ. Clearly $\delta \to 1$ as $k \to k_c$ from above, but there is little hope of obtaining a value for δ analytically except in one special case. For $K = (\sigma+1)$ we have the analytic solution, $V = \delta^{-1}$, along Δ and the condition that this curve must pass through S gives $\delta = \delta^{*} = \frac{(\gamma+1)}{2}$ which, together with $K = (\sigma+1)$, implies $k = k^{*} = \frac{2\gamma(\sigma+1)}{(\gamma+1)} + \frac{2(\gamma-1)}{(\gamma+1)}$.

For other values of k we must resort to some numerical technique for evaluating δ . The method chosen is quite straightforward and it is, in fact, similar to that used in evaluating k_c.

We produce upper and lower bounds, δ_1 and δ_0 respectively, for δ which we refine using the bisection method. The algorithm in this case is as follows: for fixed values of k, γ, σ we take the bisected value, δ_2, of our bounds on δ and we construct a part of Δ by numerically integrating equation (3.4.1) from singular point G up to $Z = Z_S$ giving $V = V_1$ at this point. If $V_1 < V_S$ then δ_2 is too large and we replace δ_1 by δ_2 whereas if $V_1 > V_S$ then δ_2 is too small and we replace δ_0 by δ_2. This is repeated until we have sufficiently fine bounds on δ.

For $k_c < k < k^*$ we already have bounds on δ in $\delta_1 = \delta^*$ and $\delta_0 = 1$. For $k > k^*$ we only have a lower bound, $\delta_0 = \delta^*$, initially and so we must produce an upper bound artificially.. We do this by choosing an incremental value, a say, of δ. Now let $\delta_1 = \delta_0 + a$ and construct that part of Δ from G up to $Z = Z_S$ using $\delta = \delta_1$ and obtain $V = V_1$. If $V_1 > V_S$ then δ_1 is too small giving a higher lower bound. In this case we replace δ_0 by δ_1 and increase δ_1 by a and so on until we find that $V_1 < V_S$ which implies that δ_1 is too large. Then we have upper and lower bounds on δ which are a apart and we can refine them as before.

Graphs of δ against k for various values of γ, σ are shown in FIGS. 19 to 21.

6.2. The inner expansions.

In the previous section we discussed the possible forms of the asymptotic solution in the sense that we determined the singular point in the plane of Z and V to which the solution curve goes and also we determined the parameter δ. We use that information in this section and produce the inner expansions which are valid for $\phi_1 = O(1)$.

The starting point is at the inner equations, (4.3.2). Now the shock path is asymptotically of the form $r = \zeta t^{\delta} + o(t^{\delta})$ and therefore $V(r) = b_0 r^{\epsilon} + o(r^{\epsilon})$, where $b_0 = \frac{2\delta}{(\gamma+1)} \zeta^{\frac{1}{\delta}}$. Thus for large r, $\phi_1 = O(1)$, we expand

$$\left.\begin{aligned}
u &= b_0 r^{\epsilon}\left\{U_0(\phi_1) + b_1 \frac{U_1(\phi_1)}{r^w} + \ldots\ldots\right\} ,\\
p &= \frac{(\gamma+1)}{2} b_0^2 r^{2\epsilon-k}\left\{P_0(\phi_1) + b_1 \frac{P_1(\phi_1)}{r^w} + \ldots\ldots\right\} ,\\
\rho &= \frac{(\gamma+1)}{(\gamma-1)} r^{-k}\left\{\rho_0(\ _1) + b_1 \frac{\rho_1(\phi_1)}{r^w} + \ldots\ldots\right\} ,\\
V(r) &= b_0 r^{\epsilon}\left\{1 + \frac{b_1}{r^w} + \ldots\ldots\right\} ,
\end{aligned}\right\} \quad (6.2.1)$$

where b_0 , b_1 , $w > 0$ are unknown constants.

The strong shock conditions give

$$\left.\begin{aligned}
&U_0(1) = P_0(1) = \rho_0(1) = 1 ,\\
&U_1(1) = 1 , P_1(1) = 2 , \rho_1(1) = 0 .
\end{aligned}\right\} \quad (6.2.2)$$

Substituting expansions (6.2.1) into equations (4.3.2) and equating coefficients of appropriate powers of r to zero gives zeroth and first order equations for U_0, ρ_0, P_0 and U_1, ρ_1, P_1 respectively.

The zeroth order equations are

$$\phi_1\left[U_0\rho_0' + \rho_0 U_0'\right] - \frac{(\gamma+1)}{(\gamma-1)}\rho_0^2 U_0' + \frac{(\sigma+\epsilon-k)}{(k-\sigma-1)}\rho_0 U_0 = 0 ,$$

$$\left.\begin{aligned}
&\rho_0 U_0 \phi_1 U_0' + \frac{\epsilon \rho_0 U_0^2}{(k-\sigma-1)} + \frac{(\gamma-1)}{2}\phi_1 P_0' - \frac{(\gamma+1)}{2}\rho_0 P_0' \\
&\qquad\qquad - \frac{(\gamma-1)(k-2\epsilon)}{2(k-\sigma-1)} P_0 = 0\ , \\
&P_0 = \rho_0^{\gamma}\, \phi_1^{\frac{-[k(\gamma-1)+2\epsilon]}{(k-\sigma-1)}}\ ,
\end{aligned}\right\} (6.2.3)$$

while the first order equations are

$$\left.\begin{aligned}
&\phi_1 \left[U_0\rho_1' + U_1\rho_0' + \rho_0 U_1' + \rho_1 U_0'\right] \\
&\quad - \frac{(\gamma+1)}{(\gamma-1)}\left[\rho_0^2 U_1' + 2\rho_0\rho_1 U_0'\right] \\
&\quad + \frac{(\sigma+\epsilon-k-w)}{(k-\sigma-1)}\left(\rho_0 U_1 + \rho_1 U_0\right) = 0\ , \\
&\phi_1\left[\rho_0 U_0 U_1' + \rho_0 U_1 U_0' + \rho_1 U_0 U_0'\right] - \frac{w\rho_0 U_0 U_1}{(k-\sigma-1)} \\
&\quad + \epsilon U_0\left[\rho_1 U_0 + 2\rho_0 U_1\right] + \frac{(\gamma-1)}{2}\phi_1 P_1' \\
&\quad - \frac{(\gamma+1)}{2}\left[\rho_0 P_1' + \rho_1 P_0'\right] - \frac{(\gamma-1)(k+w-2\epsilon)}{(k-\sigma-1)} P_1 = 0\ , \\
&P_1 = P_0\left\{\frac{\gamma\rho_1}{\rho_0} + 2\,\phi_1^{\frac{w}{(k-\sigma-1)}}\right\}.
\end{aligned}\right\} (6.2.4)$$

To find the region of validity of expansions (6.2.1) we seek series solutions of (6.2.3) and (6.2.4) in the outer limit of ϕ_1 . To help in this matter we note that, near F in the phase plane of Z and V,

$$\left.\begin{aligned} R(\lambda) &= R^{*}\left(\frac{\lambda}{\xi}\right)^{\frac{-(k-\sigma-1)}{(\delta-1)}} + \ldots\ldots , \\ V(\lambda) &= \delta^{-1} + \ldots\ldots , \end{aligned}\right\} \quad (6.2.5)$$

where R^{*} is some constant and $\lambda = \xi$ is constant, as $t \to \infty$, on the shock.

Now, for $r \to \infty$, $\frac{(\gamma+1)}{(\gamma-1)} \rho_0(\phi_1)$ and $b_0\, U_0(\phi_1)$ are equivalent to $R(\lambda)$ and $\delta\lambda^{\frac{1}{\delta}}\, V(\lambda)$ respectively. Using this information, then, expression (4.3.4) gives, in the outer limit,

$$\rho_0(\phi_1) = \frac{(\gamma-1)}{(\gamma+1)} \frac{1}{\epsilon} \phi_1 + \ldots\ldots \quad . \qquad (6.2.6)$$

Comparing (6.2.5), (6.2.6) gives

$$\lambda = \xi \left(\frac{\phi_1}{R^{*}\epsilon}\right)^{\frac{-(\delta-1)}{(k-\sigma-1)}} + \ldots\ldots ,$$

and thus

$$b_0\, U_0(\phi_1) = \delta\lambda^{\frac{1}{\delta}}\, V = \lambda^{\frac{1}{\delta}} + \ldots\ldots$$

$$= \xi^{\frac{1}{\delta}} \left(\frac{\phi_1}{R^{*}\epsilon}\right)^{\frac{-\epsilon}{(k-\sigma-1)}} + \ldots\ldots ,$$

and, since $b_0 = \frac{2\delta}{(\gamma+1)} \xi^{\frac{1}{\delta}}$, this gives

$$U_0(\phi_1) = \frac{(\gamma+1)}{2\delta}\left(\frac{\phi_1}{R^{*}\epsilon}\right)^{\frac{-\epsilon}{(k-\sigma-1)}} + \ldots\ldots , \qquad (6.2.7)$$

as $\phi_1 \to \infty$.

Consideration of equation (6.2.6) and the third of (6.2.3) leads to

$$P_0(\phi_1) = B_0^{\gamma}\, \phi_1^{\frac{-[\gamma(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}} + \ldots\ldots , \qquad (6.2.8)$$

where $B_0 = \frac{(\gamma-1)}{(\gamma+1)}\frac{1}{\epsilon}$.

Thus, guided by (6.2.6), (6.2.7), (6.2.8), we seek series solutions for U_0, ρ_0, P_0 for large ϕ_1 and we obtain

$$\left.\begin{aligned}
U_0(\phi_1) &= A_0\, \phi_1^{\frac{-\epsilon}{(k-\sigma-1)}}\left\{1 + A_1\, \phi_1^{\frac{-(\gamma-1)(\sigma+1)}{(k-\sigma-1)}} + \ldots\right\} ,\\
P_0(\phi_1) &= B_0^{\gamma}\, \phi_1^{\frac{-[\gamma(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}}\left\{1 + \gamma B_1\, \phi_1^{\frac{-\beta_1}{(k-\sigma-1)}} + \ldots\right\} ,\\
\rho_0(\phi_1) &= B_0\, \phi_1\left\{1 + B_1\, \phi_1^{\frac{-\beta_1}{(k-\sigma-1)}} + \ldots\ldots\right\} ,
\end{aligned}\right\} \qquad (6.2.9)$$

where $A_1 = \frac{[\gamma(\sigma+1)(1-\epsilon) + 2\epsilon-k]\, B_0^{\gamma-1}}{2\,\epsilon\,(\sigma+1)\, A_0^2}$,

$$\beta_1 = \min\left[1, (\gamma-1)(\sigma+1)\right] ,$$

and A_0 is undetermined by the analysis.

If $\beta_1 = 1$ then B_1 is also undetermined but can be found numerically if necessary, while if $\beta_1 = (\gamma-1)(\sigma+1)$ then

$$B_1 = \frac{-(\gamma+1)(1-\epsilon)\left[\gamma(\sigma+1)(1-\epsilon) + 2\epsilon - k\right] B_0^{\gamma}}{2\epsilon A_0^2 \left[1 - (\gamma-1)(\sigma+1)\right]} .$$

For the sake of completeness we note that, for $K = (\sigma+1)$, $\delta = \frac{(\gamma+1)}{2}$ we have an analytic solution of equations (6.2.3). It is

$$\left.\begin{aligned} U_0(\phi_1) &= \phi_1^{\frac{-1}{(\sigma+3)}} , \\ P_0(\phi_1) &= \phi_1^{\frac{-\gamma(\sigma+1)}{(\sigma+3)}} , \\ \rho_0(\phi_1) &= \phi_1 . \end{aligned}\right\} \qquad (6.2.10)$$

Turning to the first order problem we can see that equations (6.2.4) are linear and inhomogeneous and so the solution can be split up into the particular integral, corresponding to the inhomogeneous term, and a complementary function which is the general solution of the homogeneous equations and therefore includes two arbitrary constants. To analyse the first order equations we make a change of dependent variables by letting

$$U_0(\phi_1) = \phi_1^{\alpha_1} v_0(\phi_1) , \quad U_1(\phi_1) = \phi_1^{\alpha_1+\nu} v(\phi_1) ,$$

$$P_0(\phi_1) = \phi_1^{\alpha_2}\, p_0(\phi_1) \;,\; P_1(\phi_1) = \phi_1^{\alpha_2+\nu}\, p_1(\phi_1) \;,$$

$$\rho_0(\phi_1) = \phi_1 \;\; s_0(\phi_1) \;,\; \rho_1(\phi_1) = \phi_1^{1+\nu}\; s_1(\phi_1) \;,$$

where $\alpha_1 = \frac{-\epsilon}{(k-\sigma-1)}$, $\alpha_2 = \frac{-[\gamma(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}$, $\nu = \frac{w}{(k-\sigma-1)}$,

and then the first order equations become

$$\left.\begin{aligned}
&\phi_1\left[v_0 s_1' + v_1 s_0' + s_0 v_1' + s_1 v_0'\right] - \frac{(\gamma+1)}{(\gamma-1)} s_0 \phi_1 \left[s_0 v_1' + 2 s_1 v_0'\right] \\
&- \frac{(s_0 v_1 + s_1 v_0)}{(k-\sigma-1)} - \frac{(\gamma+1)\; s_0}{(\gamma-1)(k-\sigma-1)}\left[(w-\epsilon) s_0 v_1 - 2\epsilon s_1 v_0\right] = 0, \\
&\phi_1\left[s_0 v_0 v_1' + s_0 v_1 v_0' + s_1 v_0 v_0'\right] \\
&+ \frac{(\gamma-1)}{2}\,\phi_1^{\frac{-(\gamma-1)(\sigma+1)}{(k-\sigma-1)}}\left\{\phi_1 p_1' - \frac{(\gamma+1)}{(\gamma-1)} s_0 \phi_1 p_1' - \frac{(\gamma+1)}{(\gamma-1)} s_1 \phi_1 p_0'\right. \\
&- \frac{\gamma(\sigma+1)}{(k-\sigma-1)} p_1 + \frac{(\gamma+1)}{(\gamma-1)} \frac{[\gamma(\sigma+1) + 2\epsilon - k - w]}{(k-\sigma-1)} s_0 p_1 \\
&\left. + \frac{(\gamma+1)}{(\gamma-1)} \frac{[\gamma(\sigma+1) + 2\epsilon - k]}{(k-\sigma-1)} s_1 p_0\right\} = 0 \;, \\
&s_0 p_1 = \gamma s_1 p_0 + 2 s_0 p_0 \;.
\end{aligned}\right\} \quad (6.2.11)$$

Series solutions of these equations, for $\phi_1 \to \infty$, can be obtained, the leading terms in the particular integral being

$$v_1^{(1)}(\phi_1) = \frac{\epsilon}{w} A_0 B_2 + \ldots\ldots ,$$

$$s_1^{(1)}(\phi_1) = B_0 B_2 + \ldots\ldots ,$$

$$p_1^{(1)}(\phi_1) = B_0^{\gamma}\left[\gamma B_2 + 2\right] + \ldots\ldots ,$$

where B_2 is indeterminate, and those in the complementary function

$$v_1^{(2)}(\phi_1) = \frac{\epsilon}{w} A_0 B_4 + A_0 A_5 \phi_1^{-\left[\frac{1+(\gamma-1)(\sigma+1)}{(k-\sigma-1)}\right]} + \ldots\ldots ,$$

$$s_1^{(2)}(\phi_1) = B_0\left\{B_4 + B_5 \phi_1^{\frac{-1}{(k-\sigma-1)}} + \ldots\ldots\right\} ,$$

$$p_1^{(2)}(\phi_1) = \gamma B_0^{\gamma}\left\{B_4 + B_5 \phi_1^{\frac{-1}{(k-\sigma-1)}} + \ldots\ldots\right\} ,$$

where B_4 , B_5 are independent arbitrary constants and A_5 is linearly dependent upon B_5 .

The full first order solutions are thus

$$v_1(\phi_1) = v_1^{(1)}(\phi_1) + v_1^{(2)}(\phi_1) ,$$

$$s_1(\phi_1) = s_1^{(1)}(\phi_1) + s_1^{(2)}(\phi_1) ,$$

$$p_1(\phi_1) = p_1^{(1)}(\phi_1) + p_1^{(2)}(\phi_1) .$$

6.3. The breakdown of the inner solution.

It is evident that the expansions for u, p, ρ each breakdown when $\phi_1 = O(r^{k-\sigma-1})$. Thus we need to seek a different asymptotic solution in the outer region near the contact front, where the appropriate variables are ϕ and r.

Writing the inner expansions in terms of the outer variables gives the outer limit of the inner solutions:

$$u = b_0 A_0^{\frac{-\epsilon}{(k-\sigma-1)}} \left\{ \left[1 + \frac{b_1 \epsilon}{w}(B_2 + B_4)\, \phi^{\frac{w}{(k-\sigma-1)}} \right] + \frac{A_1 \phi^{\frac{-(\gamma-1)(\sigma+1)}{(k-\sigma-1)}}}{r^{(\gamma-1)(\sigma+1)}} + \frac{b_1 A_5 \phi^{\left[\frac{w-1-(\gamma-1)(\sigma+1)}{(k-\sigma-1)}\right]}}{r^{1+(\gamma-1)(\sigma+1)}} + \ldots \right\}, \quad (6.3.1)$$

$$\rho = \frac{(\gamma+1)}{(\gamma-1)} B_0 r^{-(\sigma+1)} \phi \left\{ \left[1 + b_1(B_2 + B_4)\, \phi^{\frac{w}{(k-\sigma-1)}} \right] + \frac{B_1 \phi^{\frac{-\beta_1}{(k-\sigma-1)}}}{r^{\beta_1}} + \frac{b_1 B_5 \phi^{\frac{(w-1)}{(k-\sigma-1)}}}{r} + \ldots\ldots \right\}, \quad (6.3.2)$$

$$p = \frac{(\gamma+1)}{2} b_0^2 B_0^{\gamma} r^{-\gamma(\sigma+1)} \phi^{-\left[\frac{\gamma(\sigma+1)+2\epsilon-k}{k-\sigma-1}\right]} \left\{ \left[1 + b_1(\gamma(B_2+B_4)+2)\, \phi^{\frac{w}{(k-\sigma-1)}} \right] + \frac{B_1 \phi^{\frac{-\beta_1}{(k-\sigma-1)}}}{r^{\beta_1}} + \frac{\gamma B_5 b_1 \phi^{\frac{(w-1)}{(k-\sigma-1)}}}{r} + \ldots\ldots \right\}, \quad (6.3.3)$$

where A_0, B_0, A_1, B_1, B_2, B_4, B_5, A_5, β_1, are defined in section 6.2.. These show that, near the contact front,

$$u = O(1), \quad p = O(r^{-\gamma(\sigma+1)}), \quad \rho = O(r^{-(\sigma+1)}). \quad (6.3.4)$$

6.4 The outer expansions and matching.

In this section we seek asymptotic expansions which are valid near the contact front and which match, up to the orders taken, with the inner expansions. Unfortunately a complete asymptotic solution cannot be produced but we do show that series solutions, which match with the inner solutions, can be found.

A full numerical analysis of the problem was carried out and it was found that, for the particular case chosen, the density and pressure are indeed of order $r^{-(\sigma+1)}$ and $r^{-\gamma(\sigma+1)}$ respectively, within acceptable numerical error and, moreover, the profile for p/p_s at large r is very near the corresponding asymptotic profile. Also the value of δ given by the asymptotic analysis compares most favourably with that given by the full numerical solution. The details of this are given in chapter 7. Now we can proceed with the analysis with complete confidence.

Guided by matching conditions we seek asymptotic expansions of the form

$$\left.\begin{aligned} u &= b_0\left\{u_0(\phi) + \frac{u_1(\phi)}{r^{\alpha}} + \ldots\ldots\right\} \quad, \\ p &= \frac{(\gamma+1)}{2}\, b_0^2\, r^{-\gamma(\sigma+1)}\left\{\Pi_0(\phi) + \frac{\Pi_1(\phi)}{r^{\beta}} + \ldots\ldots\right\} \quad, \\ \rho &= \frac{(\gamma+1)}{(\gamma-1)}\, r^{-(\sigma+1)}\left\{R_0(\phi) + \frac{R_1(\phi)}{r^{\beta}} + \ldots\ldots\right\} \quad, \end{aligned}\right\} \qquad (6.4.1)$$

where α and β are positive constants to be determined.

The insertion of expansions (6.4.1) into the outer equations gives

$$-\beta\frac{u_0R_1}{r^\beta} - \alpha\frac{R_0u_1}{r^\alpha} - (k-\sigma-1)\frac{(\gamma+1)}{(\gamma-1)}\left(R_0^2 + \frac{2R_0R_1}{r^\beta}\right)\left(u_0' + \frac{u_1'}{r^\alpha}\right)$$

$$-\left(R_0 + \frac{R_1}{r^\beta}\right)\left(u_0 + \frac{u_1}{r^\alpha}\right) = o(r^{-\alpha}) + o(r^{-\beta}) \quad , \qquad (6.4.2)$$

$$-\alpha\frac{R_0u_0u_1}{r^\alpha} - \gamma\frac{(\gamma-1)}{2}(\sigma+1)\ r^{-(\gamma-1)(\sigma+1)}\Pi_0$$

$$-(k-\sigma-1)\frac{(\gamma+1)}{2}\ r^{-(\gamma-1)(\sigma+1)}\ R_0\Pi_0' = o(r^{-\alpha})$$

$$+\ o(r^{-(\gamma-1)(\sigma+1)}) \quad , \qquad (6.4.3)$$

$$\Pi_0 + \frac{\Pi_1}{r^\beta} = R_0^\gamma\left(1 + \frac{\gamma R_1}{R_0\ r^\beta}\right) W^2\left(\phi^{\frac{-1}{(k-\sigma-1)}}\right)\phi^{\frac{-[k(\gamma-1)+2\epsilon]}{(k-\sigma-1)}} , \qquad (6.4.4)$$

with W(r) defined in

$$V(r) = b_0\ r^\epsilon\ W(r) .$$

Terms of O(1) in (6.4.2) give

$$\frac{u_0'}{u_0} = \frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)\ R_0} , \qquad (6.4.5)$$

which gives, on integrating and using $b_0u_0(1) = 1$,

$$\ln(b_0u_0) = \frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)}\int_1^{\phi}\frac{dy}{R_0(y)} \quad . \tag{6.4.6}$$

Now suppose $\alpha < \beta$, then $O(r^{-\alpha})$ terms in (6.4.2) give

$$-\alpha R_0u_1 - (k-\sigma-1)\frac{(\gamma+1)}{(\gamma-1)} R_0^2 \, u_1' - R_0u_1 = 0 \quad ,$$

which has a general solution for u_1 , on using (6.4.5),

$$u_1 = C_1 \, u_0^{(1+\alpha)} \quad , \tag{6.4.7}$$

C_1 being an arbitrary constant. The boundary condition on the contact front implies that $C_1 = 0$ and hence $u_1 = 0$ if $\alpha < \beta$.

Now suppose that $\beta < \alpha$, then $O(r^{-\beta})$ terms in (6.4.2) give the fact that $R_1(\phi)$ is arbitrary if $\beta = 1$ or zero for any other $\beta < \alpha$.

The other possibility is that $\alpha = \beta$, and then we have

$$(k-\sigma-1)\frac{(\gamma+1)}{(\gamma-1)}R_0^2u_1' + (\alpha-1)R_1u_0 + (\alpha+1)R_0u_1 = 0 \; . \tag{6.4.8}$$

Turning now to (6.4.3), it is clear that $\alpha \geqslant (\gamma-1)(\sigma+1)$ if $u_1 \neq 0$. If $\alpha > (\gamma-1)(\sigma+1)$, then $O(r^{-(\gamma-1)(\sigma+1)})$ terms give

$$\frac{\Pi_0'}{\Pi_0} = \frac{-\gamma(\sigma+1)(\gamma-1)}{(\gamma+1)(k-\sigma-1)\,R_0} = \gamma(\sigma+1)\,\frac{u_0'}{u_0} \quad .$$

This can be integrated to give

$$\Pi_0 = c_0^{*}(b_0u_0)^{\gamma(\sigma+1)} \quad . \tag{6.4.9}$$

If $\alpha = (\gamma-1)(\sigma+1)$ then we have

$$\Pi_0' + \frac{\gamma(\sigma+1)(\gamma-1)\Pi_0}{(\gamma+1)(k-\sigma-1)R_0} = \frac{-2(\gamma-1)(\sigma+1)u_0u_1}{(\gamma+1)(k-\sigma-1)} \quad , \tag{6.4.10}$$

which is linear in Π_0 and so the solution will consist of a particular integral and the complementary function, which is displayed in (6.4.9) .

An examination of (6.4.4) gives

$$\left.\begin{aligned} \Pi_0 &= R_0^{\gamma}\,\phi^{\frac{-[k(\gamma-1)+2\epsilon]}{(k-\sigma-1)}}\,W^2\!\left(\phi^{\frac{-1}{(k-\sigma-1)}}\right) \quad , \\ \Pi_1 &= \frac{\gamma\Pi_0 R_1}{R_0} \quad . \end{aligned}\right\} \tag{6.4.11}$$

It can be seen that, up to the order taken, we cannot produce a closed set of equations unless we relax the boundary on u_1 or the condition $\alpha = (\gamma-1)(\sigma+1)$. Taking the first of these, we are implying that there may be yet another layer which is adjacent to the contact front. Consider equations (6.4.5), (6.4.7), (6.4.10) and the first of (6.4.11) . These present a closed set of equations but a series solution for small ϕ reveals a mismatch with the inner solution. As for the second choice we may rule this out for similar reasons.

It seems, therefore, that a complete asymptotic solution to the problem is unobtainable because of this inability to close the set of equations. In spite of this difficulty we may produce series solutions for small ϕ which will match with the inner solution.

It is interesting to note that this difficulty exists in other asymptotic solutions in gasdynamics. In the unsteady expansion of an initially uniform quiescent gas into a vacuum Hubbard (1967) noted that any asymptotic density profile, to zeroth order, satisfied the continuity equation. Grundy (1969), on examining a similar problem, found that the zeroth order velocity term was, according to the analysis, an arbitrary function of ψ, the particle path function. The flow becomes 'inertia dominated' at large times. It may be seen that his equations (2.8a,b) are similar to our equations (6.4.2), (6.4.3), (6.4.4). In his equation (2.10) for the zeroth order particle number density Grundy introduces $B(\psi)$, an arbitrary function of integration. If, in expansions (6.4.1), we had allowed R_0 to be a function of r also we would reproduce Grundy's equation (2.10) but with R_0 replacing N_0. Instead of this we have R_1 arbitrary if it is the coefficient function of r^{-1}. It can be seen that both descriptions are identical and we associate our $R_1(\phi)r^{-1}$ with Grundy's $B(\psi)r^{-1}$. Another situation where this difficulty arises is in the far downstream solution of the steady axisymmetric expansion of an inviscid, monatomic perfect gas from an orifice into a vacuum. This problem is discussed by Grundy (1969) who gives a more detailed explanation in his Ph.D. thesis. In this case the asymptotic number density is of the form $n_0(\Theta)\, r^{-2} + \ldots\ldots$, where Θ is the angle from the centreline of the jet and r is the distance from some point near to the orifice, and Grundy shows that n_0 is an arbitrary function of Θ. The conclusion we draw from all of this is that the flow near the contact front becomes 'inertia dominated' at large times and the outer solution, specifically $u_0(\phi)$ and $R_1(\phi)$, is determined by the full continuum solution and therefore by the initial conditions.

To show that a solution of the outer equations exists and matches with the inner solution we let, first of all,

$$\left.\begin{aligned} u_0(\phi) &= A_0^* \phi^{\alpha_0^*}\left\{1 + A_1^* \phi^{\alpha_1^*} + \ldots\ldots\right\} \quad , \\ R_0(\phi) &= B_0^* \phi^{\beta_0^*}\left\{1 + B_1^* \phi^{\beta_1^*} + \ldots\ldots\right\} \quad . \end{aligned}\right\} \quad (6.4.12)$$

A necessary condition for $\alpha_0^* \neq 0$ is that $\beta_0^* = 1$ and this is essential for matching of density anyway. Taking this value of β_0^* and using (6.4.12) in (6.4.6) gives

$$\alpha_0^* \ln\phi + \ln(b_0 A_0^*) + A_1^* \phi^{\alpha_1^*} =$$

$$\frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)B_0^*}\left\{\ln\phi + \int_1^{\phi}\left[\frac{B_0^*}{R_0(y)} - \frac{1}{y}\right]dy + \ldots\ldots\right\}$$

$$= \frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)B_0^*}\left\{\ln\phi + \int_0^1\left[\frac{1}{\phi} - \frac{B_0^*}{R_0(\phi)}\right]d\phi\right.$$

$$\left. - \frac{B_1^*}{\beta_1^*}\phi^{\beta_1^*} + \ldots\ldots\right\} \quad ,$$

which then implies

$$\alpha_0^* = \frac{-(\gamma-1)}{(\gamma+1)(k-\sigma-1)B_0^*} \quad ,$$

$$\left.\begin{aligned}
\ln(b_0 A_0^{*}) &= \alpha_0^{*}\int_0^1 \left[\frac{1}{\phi} - \frac{B_0^{*}}{R_0(\phi)}\right] d\phi \quad , \\
\alpha_1^{*} &= \beta_1^{*} \quad , \\
A_1^{*} &= -\frac{\alpha_0^{*}}{\beta_1^{*}} B_1^{*} \quad .
\end{aligned}\right\} \qquad (6.4.13)$$

These series expansions match, up to the first term, with the inner solution if we set

$$\left.\begin{aligned}
B_0^{*} &= B_0 = \frac{(\gamma-1)}{(\gamma+1)}\frac{1}{\epsilon} \quad , \\
\alpha_0^{*} &= \frac{-\epsilon}{(k-\sigma-1)} \quad , \\
A_0^{*} &= A_0 \quad .
\end{aligned}\right\} \qquad (6.4.14)$$

The first of equations (6.4.13) is consistent with (6.4.14) and the second gives

$$\ln(b_0 A_0) = \frac{-\epsilon}{(k-\sigma-1)}\int_0^1 \left[\frac{1}{\phi} - \frac{B_0}{R_0(\phi)}\right] d\phi \quad . \qquad (6.4.15)$$

Consider, now, the first of equations (6.4.11) and let

$$V(r) = b_0 \, r^{\epsilon}\left\{1 + \frac{b_1}{r^{W}} + \dots\dots\right\} \quad ,$$

$$\Pi_0(\phi) = B_0^{\gamma}\ \phi^{\frac{-[\gamma(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}}\left\{1 + C_1^{*}\phi^{\beta_1^{*}} + \ldots\ldots\right\} ,$$

then we have

$$1 + C_1^{*}\phi^{\beta_1^{*}} = \left\{1 + \gamma B_1^{*}\phi^{\beta_1^{*}}\right\}\left\{1 + 2b_1\phi^{\frac{w}{(k-\sigma-1)}}\right\} + \ldots\ldots .$$

For a successful match we must have

$$\beta_1^{*} = \frac{w}{(k-\sigma-1)} \quad \text{and then} \quad C_1^{*} = \gamma B_1 + 2b_1$$

and also from (6.4.13)

$$\alpha_1^{*} = \frac{w}{(k-\sigma-1)} \quad \text{and} \quad A_1^{*} = \frac{\epsilon}{w}\ B_1^{*} .$$

If we now examine equation (6.4.10) we find that the leading term in the complementary function is

$$\Pi_0^{(2)} = C_0^{*}(b_0A_0)^{\gamma(\sigma+1)}\ \phi^{\frac{-\gamma(\sigma+1)\epsilon}{(k-\sigma-1)}} + \ldots\ldots ,$$

which has no term to match with in the inner solution and so we take $C_0^{*} = 0$. The required solution of (6.4.10) , therefore, is described by its particular integral and so we set

$$u_1 = A_2^{*}\ \phi^{\alpha_2^{*}} + \ldots\ldots ,$$

which, together with the first terms of Π_0 , u_0 , R_0 , gives

$$- \frac{\left[\gamma(\sigma+1) + 2\epsilon - k\right]}{(k-\sigma-1)} B_0^{\gamma} \phi^{-\frac{\left[(\gamma-1)(\sigma+1)+2\epsilon\right]}{(k-\sigma-1)}}$$

$$+ \frac{\gamma(\sigma+1)\epsilon}{(k-\sigma-1)} B_0^{\gamma} \phi^{-\frac{\left[(\gamma-1)(\sigma+1)+2\epsilon\right]}{(k-\sigma-1)}}$$

$$= - \frac{2(\gamma-1)(\sigma+1)}{(\gamma+1)(k-\sigma-1)} A_0 A_2^{*} \phi^{\alpha_2^{*} - \frac{\epsilon}{(k-\sigma-1)}} + \ldots\ldots \quad .$$

This yields $\alpha_2^{*} = -\left\{\frac{(\gamma-1)(\sigma+1)+\epsilon}{(k-\sigma-1)}\right\}$,

$$A_2^{*} = \frac{(\gamma+1)\left[\gamma(\sigma+1)(1-\epsilon) + 2\epsilon - k\right]}{2(\gamma-1)(\sigma+1)\, A_0} B_0^{\gamma} \quad .$$

Regarding first order terms in density and pressure, if $\beta = 1$, then R_1 is arbitrary and Π_1 can only be expressed in terms of R_1 whereas, if $\beta = \alpha = (\gamma-1)(\sigma+1)$, equation (6.4.8) accepts a series expansion

$$R_1 = - \frac{B_0 A_2^{*}(\gamma-1)(\sigma+1)(1-\epsilon)}{\epsilon A_0 \left[1 - (\gamma-1)(\sigma+1)\right]} \phi^{\frac{k-\gamma(\ +1)}{(k-\sigma-1)}} + \ldots\ldots \quad ,$$

and the second of (6.4.11) then gives

$$\Pi_1 = - \frac{\gamma B_0^{\gamma} A_2^{*}(\gamma-1)(\sigma+1)(1-\epsilon)}{\epsilon A_0 \left[1 - (\gamma-1)(\sigma+1)\right]} \phi^{-\frac{\left[(2\gamma-1)(\sigma+1)+2\epsilon-k\right]}{(k-\sigma-1)}} + \ldots\ldots \quad .$$

To summarise the solutions for $(\gamma-1)(\sigma+1) < 1$, as $\phi \to 0$, we have

$$\left.\begin{aligned}
u &= b_0\left\{ A_0\,\phi^{\frac{-\epsilon}{(k-\sigma-1)}}\left[1 + \frac{\epsilon}{w}B_1^{*}\,\phi^{\frac{w}{(k-\sigma-1)}}\right] + \frac{A_2^{*}\,\phi^{-\frac{[(\gamma-1)(\sigma+1)+\epsilon]}{(k-\sigma-1)}}}{r^{(\gamma-1)(\sigma+1)}} + \ldots\right\},\\
p &= \frac{(\gamma+1)}{2}\,b_0^2\,r^{-\gamma(\sigma+1)}\left\{ B_0^{\gamma}\,\phi^{-\frac{[\gamma(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}}\left[1+(\gamma B_1^{*}+2b_1)\,\phi^{\frac{w}{(k-\sigma-1)}}\right]\right.\\
&\quad\left. - \frac{\gamma B_0^{\gamma}A_2^{*}(\gamma-1)(\sigma+1)(1-\epsilon)\,\phi^{-\frac{[(2\gamma-1)(\sigma+1)+2\epsilon-k]}{(k-\sigma-1)}}}{\epsilon\,A_0\left[1-(\gamma-1)(\sigma+1)\right]\,r^{(\gamma-1)(\sigma+1)}} + \ldots\ldots\right\},\\
\rho &= \frac{(\gamma+1)}{(\gamma-1)}\,r^{-(\sigma+1)}\left\{ B_0\,\phi\left[1+B_1^{*}\,\phi^{\frac{w}{(k-\sigma-1)}}\right]\right.\\
&\quad\left. - \frac{B_0A_2^{*}(\gamma-1)(\sigma+1)(1-\epsilon)\,\phi^{\frac{k-\gamma(\sigma+1)}{(k-\sigma-1)}}}{\epsilon\,A_0\left[1-(\gamma-1)(\sigma+1)\right]\,r^{(\gamma-1)(\sigma+1)}} + \ldots\ldots\right\}.
\end{aligned}\right\} \quad (6.4.16)$$

Comparing (6.4.16) with (6.3.1), (6.3.2), (6.3.3) shows that we have a match up to this order if

$$B_1^{*} = b_1\ (B_2 + B_4) \quad , \qquad (6.4.17)$$

$A_2^{*} = A_0\ A_1$, which is correct.

If $(\gamma-1)(\sigma+1) > 1$ then the first order terms in density and pressure are $O(r^{-1})$. Although R_1 is indeterminate we can say that a solution exists which will match with the inner expansions and this solution must be of the form

$$R_1 = B_0 \phi^{1-\frac{1}{(k-\sigma-1)}} \left\{ B_1 + b_1 B_5 \phi^{\frac{w}{(k-\sigma-1)}} + \ldots\ldots \right\} ,$$

$$\pi_1 = \gamma B_0^{\gamma} \phi^{-\frac{[\gamma(\sigma+1)+2\epsilon-k-1]}{(k-\sigma-1)}} \left\{ B_1 + b_1 B_5 \phi^{\frac{w}{(k-\sigma-1)}} + \ldots\ldots \right\} .$$

These forms satisfy the second of equations (6.4.11) and also match with appropriate terms in the inner solution.

It is evident that the outer solution contains at least two degrees of freedom, namely $u_0(\phi)$ and $R_1(\phi)$, and a complete asymptotic solution to the outer equations is not possible. This should not be unexpected since in our problem the gas velocity, density and pressure near the contact front are of the same order as the corresponding variables in the unsteady expansion into vacuum. In spite of this indeterminate nature we have made a certain amount of progress. We have set down a qualitative description of the solution to the outer equations which matches, up to the order taken, with the inner expansions. If we knew $u_0(\phi)$ then matching with the zeroth order inner solution would give a means of calculating b_0 , while matching with the first order inner solution would produce an eigenvalue problem for w. This would then be solved by choosing w so that equation (6.4.17) is satisfied.

We conclude here that the region near the contact front becomes 'inertia dominated' and the contact front, at large times, has no zeroth order influence on the motion of the shock.

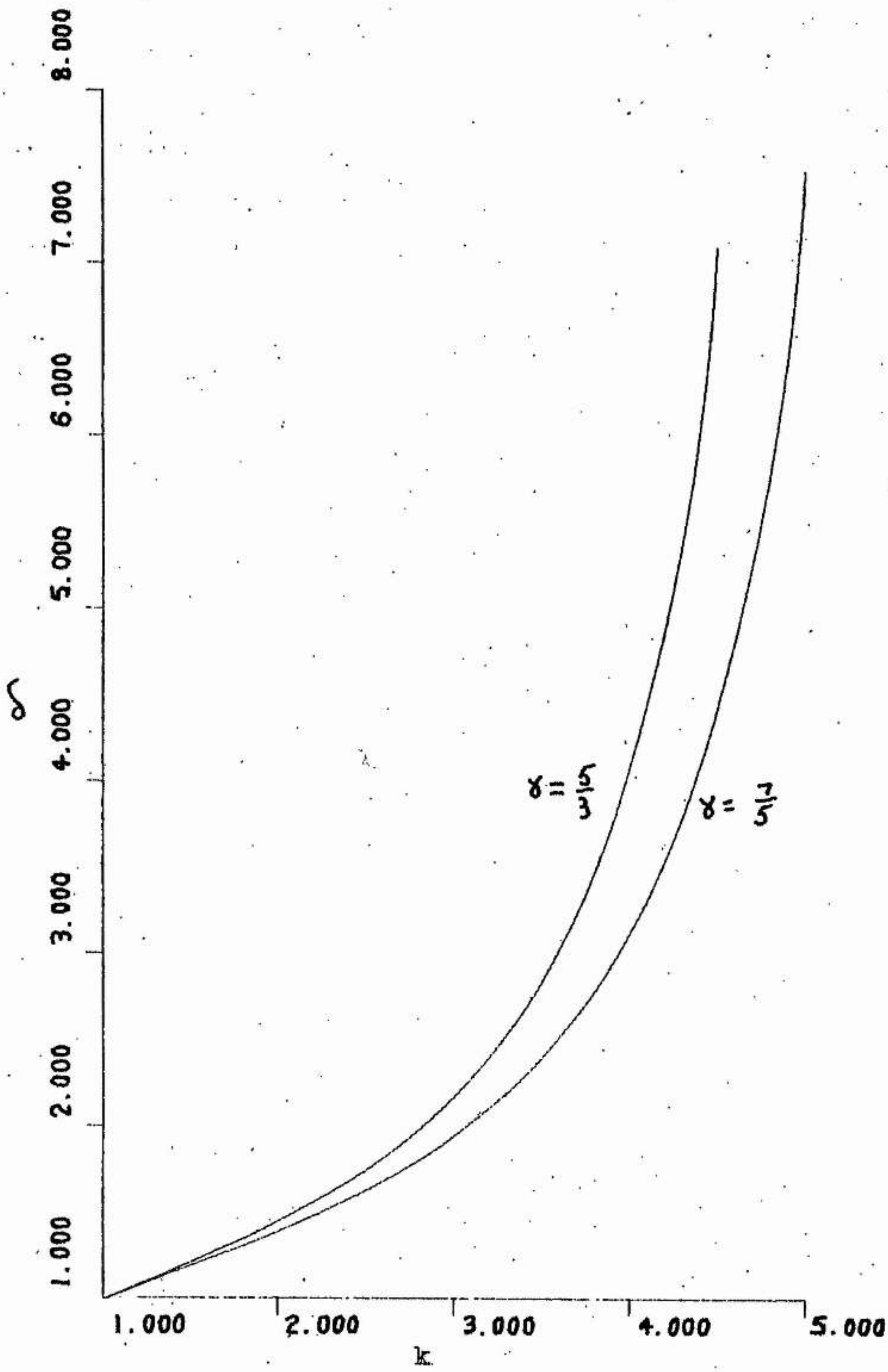


FIG. 19. $\sigma = 0$.

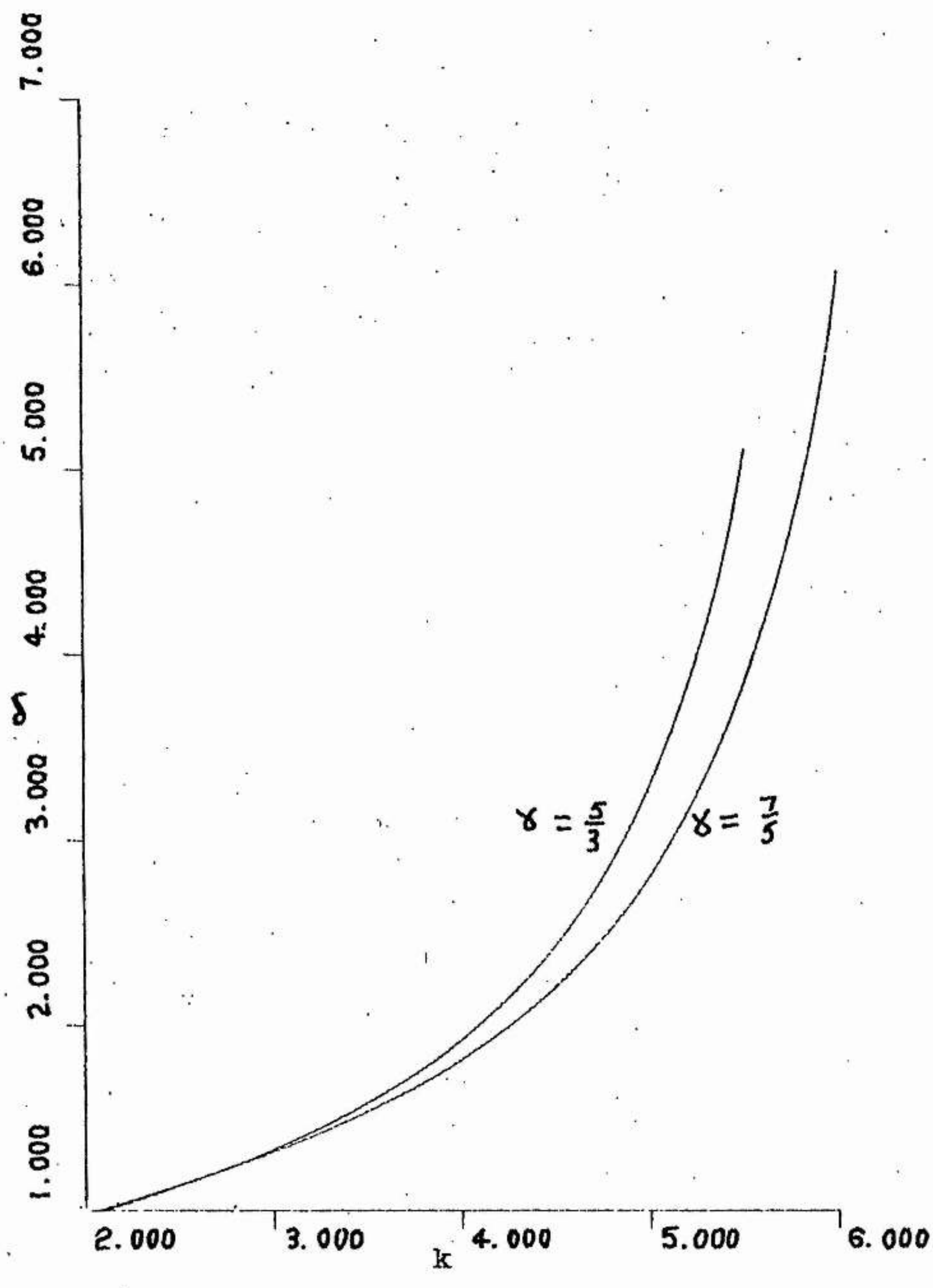


FIG. 20. $\sigma = 1$.

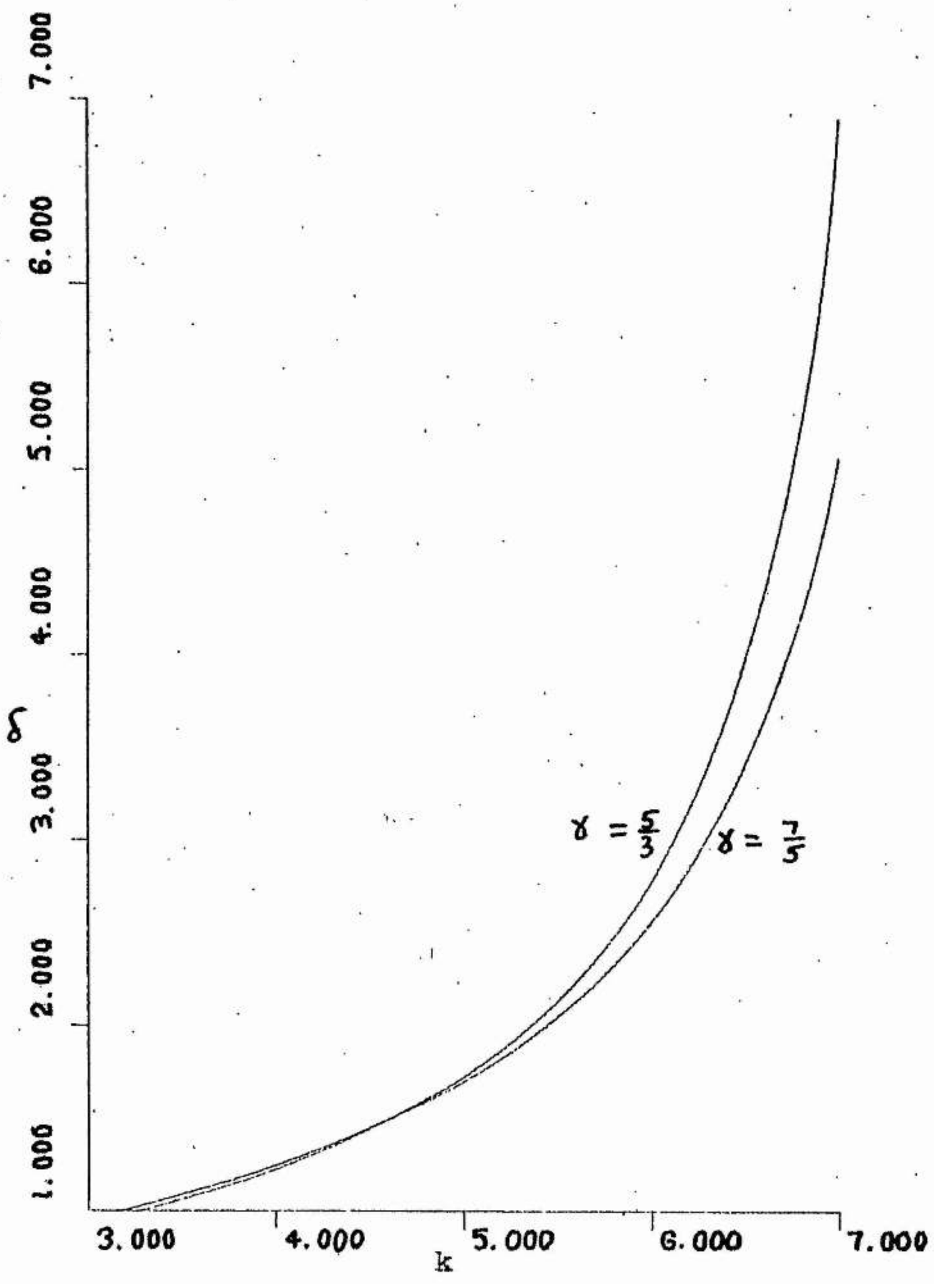


FIG. 21. $\sigma = 2$.

7. Numerical solutions for finite time.

7.1. Introduction.

We apply the method of backward drawn characteristics to the Lagrangian equations and use the Courant-Friedrichs-Lewy stability criterion. This method, devised by Hartree, is modified to calculate values of the gas variables near the shock boundary in the solution in terms of the outer variables. The application of a finite difference scheme to the characteristic equations results in a set of non-linear algebraic equations which are solved by a converging linear iterative scheme. A linear scheme was chosen in preference to a super-linear scheme for simplicity and because of the small error in the initial estimate of the solution to the algebraic equations. The added complexity of any super-linear scheme more than outweighs the saving in the number of iterations, if we are interested in the computational time factor.

Two problems were solved; one each for $k < k_c$ and $k > k_c$ and we compare certain features of these solutions with results obtained from the corresponding asymptotic analysis.

7.2. The characteristic equations.

Equations (2.3.6), written in characteristic form, are

$$\left.\begin{aligned} & dp \pm \rho a \, du + \frac{\gamma \sigma p u}{(u \pm a)} \, dr = 0 \quad , \\ & \text{on } \frac{dr}{dt} = u \pm a \quad , \\ & \text{and } d(p\rho^{-\gamma}) = 0 \quad , \\ & \text{on } \frac{dr}{dt} = u \quad , \end{aligned}\right\} \qquad (7.2.1)$$

where $a^2 = \gamma p \rho^{-1}$.

Whenever there is a choice of sign, as in the first two of equations (7.2.1), the upper sign will give the equation along C_+ , the positive characteristic, while the lower sign gives the equation along C_- , the negative characteristic. The second of equations (7.2.1) are the differential equations for the characteristic curves, C_+ and C_- , while the first of these equations are the respective compatibility conditions. The fourth of these equations, although strictly describing a characteristic curve, is the equation for the streamlines and the third of equations (7.2.1) states the conservation of entropy along a streamline.

We reintroduce the particle path function or Lagrangian variable, ψ, defined by (4.2.1), and then we let

$$\phi = \left\{ \begin{array}{ll} (\sigma+1-k)\psi , & \text{for } k < (\sigma+1) , \\ \psi , & \text{for } k = (\sigma+1) , \\ \dfrac{(k-\sigma-1)\psi}{1-(k-\sigma-1)\psi} , & \text{for } k > (\sigma+1) , \end{array} \right\} \tag{7.2.2}$$

$$y = \left\{ \begin{array}{ll} r^{(\sigma+1-k)} - 1 , & \text{for } k < (\sigma+1) , \\ \ln r , & \text{for } k = (\ +1) , \\ r^{(k-\sigma-1)} - 1 , & \text{for } k > (\sigma+1) , \end{array} \right\} \tag{7.2.3}$$

$$\theta = \phi\, y^{-1} \quad \text{for all } k . \tag{7.2.4}$$

Here ϕ and θ are related to the outer and inner variable respectively of chapter 4 .

The relationships (7.2.2), (7.2.3) have been chosen so that the flow regions in the ϕ -y plane are the same for all k, γ, σ

and then so too are those in the Θ-y plane. These regions are

$$\left.\begin{aligned} & 0 \leq \phi \leq y \quad , \quad 0 \leq y < \infty \; , \\ \text{and} \quad & 0 \leq \Theta \leq 1 \quad , \quad 0 \leq y < \infty \; . \end{aligned}\right\} \tag{7.2.5}$$

We also define

$$\left.\begin{aligned} p &= \frac{(\gamma+1)}{2} \, r^{-k} \, P \quad , \\ \rho &= \frac{(\gamma+1)}{(\gamma-1)} \, r^{-k} \, R \quad , \\ \text{and} \quad V_1(r) &= \frac{(\gamma+1)}{2} \, V(r) \quad . \end{aligned}\right\} \tag{7.2.6}$$

Using these new definitions, the characteristic equations become

$$\left.\begin{aligned} & dP \pm \frac{2}{(\gamma-1)} \, Ra \, du + \frac{\gamma P\left[\frac{\sigma u}{(u \pm a)} - \frac{k}{\gamma}\right]}{D(y)} \, dy = 0 \quad , \\ \text{on} \quad & \frac{d\phi}{dy} = \pm \frac{Ra \; E(\phi, y)}{u \pm a} \quad , \\ \text{or} \quad & \frac{d\Theta}{dy} = \frac{1}{y}\left[\frac{d\phi}{dy} - \Theta\right] \quad , \\ & d\left[PR^{-\gamma} f(y)\right] = 0 \quad , \\ \text{on} \quad & \frac{d\phi}{dy} = 0 \quad , \\ \text{or} \quad & \frac{d\Theta}{dy} = -\frac{\Theta}{y} \quad , \end{aligned}\right\} \tag{7.2.7}$$

$$\text{where} \quad D(y) = \begin{cases} (\sigma+1-k)(1+y) & , \text{ for } k < (\sigma+1) \; , \\ 1 & , \text{ for } k = (\sigma+1) \; , \\ (k-\sigma-1)(1+y) & , \text{ for } k > (\sigma+1) \; , \end{cases} \tag{7.2.8}$$

$$E(\phi,y) = \begin{cases} 1 & , \text{ for } k \leq (\sigma+1) \; , \\ \left(\frac{1+\phi}{1+y}\right)^2 = \left(\frac{1+\theta y}{1+y}\right)^2 & , \text{ for } k > (\sigma+1) \; , \end{cases} \tag{7.2.9}$$

$$f(y) = \begin{cases} (1+y)^{\frac{k(\gamma-1)}{(\sigma+1-k)}} & , \text{ for } k < (\sigma+1) \; , \\ e^{k(\gamma-1)y} & , \text{ for } k = (\sigma+1) \; , \\ (1+y)^{\frac{k(\gamma-1)}{(k-\sigma-1)}} & , \text{ for } k > (\sigma+1) \; , \end{cases} \tag{7.2.10}$$

and $a^2 = \frac{\gamma(\gamma-1)}{2} P R^{-1}$.

The boundary conditions (2.3.7), (2.3.8) become

$$u = 1 \quad \text{on } \phi = \theta = 0 \; ,$$

$$u = V \; , \quad P = V^2 \; , \quad R = 1 \quad \text{on } \phi = y \text{ or } \theta = 1 \; ,$$

while the initial conditions, at $\phi = y = 0$, are the locally plane solutions

$$u = P = R = 1 \; .$$

7.3. Hartree's method and boundary modifications.

Hartree's method has the advantage of producing solutions on a regular mesh, thereby dispensing with the need to store the coordinates of the mesh points in the memory.

Fig. 22. helps to illustrate this method as applied to our problem

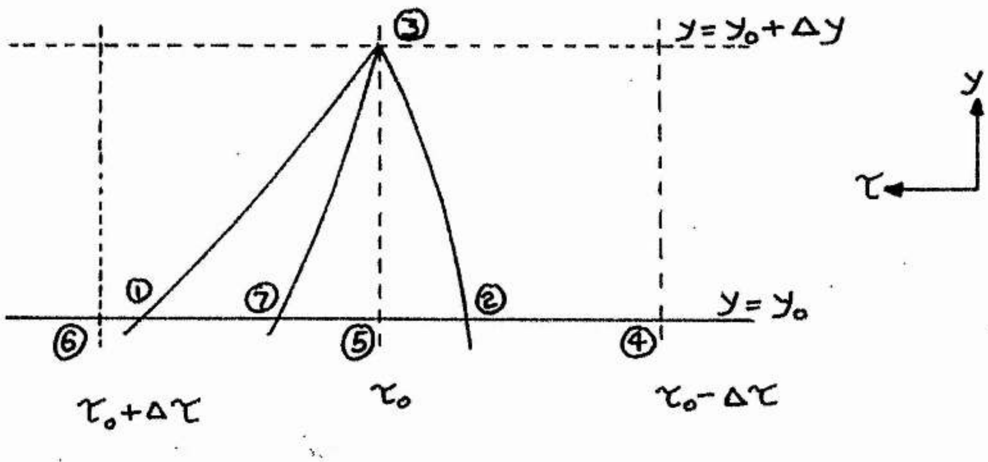


FIG. 22.

Suppose that, at some points along the line $y = y_0$, we have numerical values of the gas variables u, P, R . The object is to find corresponding values at points along the line $y = y_0 + \Delta y$. We take a specific value, τ_0 , of τ (corresponding to ϕ or Θ) on $y = y_0 + \Delta y$, that is point 3 in FIG. 22. Then we construct the three characteristic curves which pass through this point back to the line $y = y_0$ and apply the relevant compatibility condition along each characteristic. In order to carry this out in an approximate manner we apply a second order differencing scheme to all of equations (7.2.7) except the last three which can be treated analytically. In this differencing scheme we replace the differential of a quantity by the difference of its values at each end of the relevant characteristic and we replace each coefficient by the average of its values at each end of this characteristic. Values of the gas variables at points 1 , 2 , 7 are obtained by second order interpolation using values at points 4 , 5 , 6 . This now presents a set of non-linear algebraic equations which must be solved by some iterative scheme. We use a linear scheme for the reasons given in the introduction to this chapter. The sequence in which the next set of iterates is obtained simplifies the method even more since the first M, say, of the equations are independent of the rest when using the iterative scheme.

The variables which we need to solve for are τ_1 , τ_2 , τ_7 , u_1 , P_1 , R_1 , u_2 , P_2 , R_2 , P_7 , R_7 , u_3 , P_3 , R_3 and of these fourteen variables only the last three are necessary to display the solution to the problem. Nevertheless we must calculate all of them. Three of these, namely τ_7 , P_7 , R_7 , are constant for a particular point 3 and can therefore be removed from the iterative scheme, that is

$$\phi_7 = \tau_0$$

$$\text{or} \quad \theta_7 = \frac{\tau_0(y_0+\Delta y)}{y_0} \quad ,$$

and P_7 , R_7 are found by interpolation. The method used to calculate the approximate value, $g^*(x)$, of a function $g(x)$, given the numbers $g_0 = g(x_0)$, $g_1 = g(x_1)$, $g_2 = g(x_2)$, is

$$g^*(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} g_0 + \frac{(x_1-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} g_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} g_2 \quad . \qquad (7.3.1)$$

Now for initial estimates, denoted by affix (o), of the eleven unknowns we use values at point (5), that is $u_3^{(o)} = u_5$, etc., and we produce new iterates, denoted by affix (1). This is done by the following algorithm:

$$R_3^{(1)} = R_7 \left(\frac{P_3^{(o)}}{P_7}\right)^{\frac{1}{\gamma}} \left[\frac{f(y_0+\Delta y)}{f(y_0)}\right]^{\frac{1}{\gamma}} \quad ,$$

$$\left.\begin{aligned} \tau_1^{(1)} &= \tau_3 - \frac{\Delta y}{2}\left[\left(\frac{d\tau}{dy}\right)_{\overline{3}}^{(o)} + \left(\frac{d\tau}{dy}\right)_{\overline{1}}^{(o)}\right] \quad , \\ \tau_2^{(1)} &= \tau_3 - \frac{\Delta y}{2}\left[\left(\frac{d\tau}{dy}\right)_{3}^{+(o)} + \left(\frac{d\tau}{dy}\right)_{2}^{+(o)}\right] \quad , \end{aligned}\right\} \qquad (7.3.2)$$

where $\left(\frac{d\tau}{dy}\right)_{\overline{3}}^{(o)}$ denotes the gradient of the negative characteristic at point (3) evaluated using the current values of the variables (including $R_3^{(1)}$), and $f(y)$ is given by (7.2.10)

Next we use the interpolation scheme (7.3.1) to find values of u, P, R at $\tau = \tau_1^{(1)}, \tau_2^{(2)}$ using points (4), (5), (6) and we denote these by $u_1^{(1)}$, etc..

The final two equations are the compatibility conditions which must be adhered to along the C_+ and C_- characteristic curves. The two algebraic equations approximating these conditions are linear in $u_3^{(1)}$, $P_3^{(1)}$ and can be solved quite easily. These equations are

$$P_3^{(1)} - P_1^{(1)} - \frac{1}{(\gamma-1)}\left[R_3^{(1)} a_3 + R_1^{(1)} a_1^{(1)}\right](u_3^{(1)} - u_1^{(1)})$$

$$+ \frac{\gamma\,\Delta y}{2}\left\{\frac{P_3^{(0)}}{D(y_0+\Delta y)}\left[\frac{\sigma u_3^{(0)}}{u_3^{(0)}-a_3} - \frac{k}{\gamma}\right] + \frac{P_1^{(1)}}{D(y_0)}\left[\frac{\sigma u_1^{(1)}}{u_1^{(1)} - a_1^{(1)}} - \frac{k}{\gamma}\right]\right\} = 0 \quad , \tag{7.3.3}$$

$$P_3^{(1)} - P_2^{(1)} + \frac{1}{(\gamma-1)}\left[R_3^{(1)} a_3 + R_2^{(1)} a_2^{(1)}\right](u_3^{(1)} - u_2^{(1)})$$

$$+ \frac{\gamma\,\Delta y}{2}\left\{\frac{P_3^{(0)}}{D(y_0+\Delta y)}\left[\frac{\sigma u_3^{(0)}}{u_3^{(0)}+a_3} - \frac{k}{\gamma}\right] + \frac{P_2^{(1)}}{D(y_0)}\left[\frac{\sigma u_2^{(1)}}{u_2^{(1)} + a_2^{(1)}} - \frac{k}{\gamma}\right]\right\} = 0 \quad , \tag{7.3.4}$$

where $D(y)$ is given by (7.2.8),

$$a_k^{(1)} = \left[\frac{\gamma(\gamma-1)}{2}\,\frac{P_k^{(1)}}{R_k^{(1)}}\right]^{\frac{1}{2}} \quad ,$$

and
$$a_3 = \left[\frac{\gamma(\gamma-1)}{2}\,\frac{P_3^{(0)}}{R_3^{(1)}}\right]^{\frac{1}{2}} \quad .$$

Now we have new iterates and if we do not have sufficient accuracy we replace the old by the new iterates, that is let $u_1^{(0)} = u_1^{(1)}$, etc. , and repeat the above until convergence is achieved. A good test for convergence is to measure the length of the difference of the vectors denoting old and new iterates and seeing how small it is. That is, assume convergence is achieved when

$\sum_{i=1}^{11} (x_i^{(0)} - x_i^{(1)})^2 < \epsilon^2$, where ϵ is some small quantity.

When point (3) lies on the contact front, $\phi = \theta = 0$, only the C_- characteristic and the streamline are used since we have the boundary condition $u = 1$. The streamline coincides with the contact front and so we can find P_7 , R_7 without interpolation. The number of algebraic equations is reduced to seven thus simplifying the method here.

If point (3) lies on the shock boundary, then the C_+ characteristic only is used and we have, in effect, two boundary conditions in $R_3 = 1$, $P_3 = u_3^2$. The number of algebraic equations is reduced to five.

In the ϕ-y coordinate system the shock boundary has a constant slope and the range of ϕ is increased as we progress with the integration. Since we are trying to produce numerical solutions on a rectangular mesh, of element size $\Delta y \times \Delta\phi$, this increases the number of mesh points on lines of constant y. The size of Δy is arrived at by stability considerations in the next section and it is chosen so that $M\Delta y = \Delta\phi$, where M is an integer, and this means that, if we have N equidistant points on the line $y = y_0$, then we will have N+1 such points on the line $y = y_0 + M\Delta y = y_0 + \Delta\phi$. For values of y between this y_0 and $y_0 + \Delta\phi$ there will be a space between the shock and the adjacent point of length less than $\Delta\phi$. This adjacent point will be the N'th point, the first being on the contact front. When point (3) is this N'th point, then the C_- curve might reach the shock, $\phi = y$, before it reaches the previous line of constant y, the point where it intersects this previous line then being outwith the region (7.2.5). If we used this outside point as point (1) we would have to extrapolate to find u_1, etc..

This is a state of affairs which is likely to introduce instability to the solution and thus we must devise some way to avoid it without losing the accuracy of the method. The obvious choice for point (1) is the intersection of the negative characteristic under discussion and the shock boundary. Whether this curve reaches the shock first or not is determined by approximately solving a locally linearised version of the characteristic equation. If this approximate solution indicates that point (1) is on the shock we proceed with an algorithm differing from the one above in the treatment of the C_- characteristic with the corresponding compatibility condition and also in the evaluation of u_1 , P_1 , R_1 .

FIG. 23. shows the picture now.

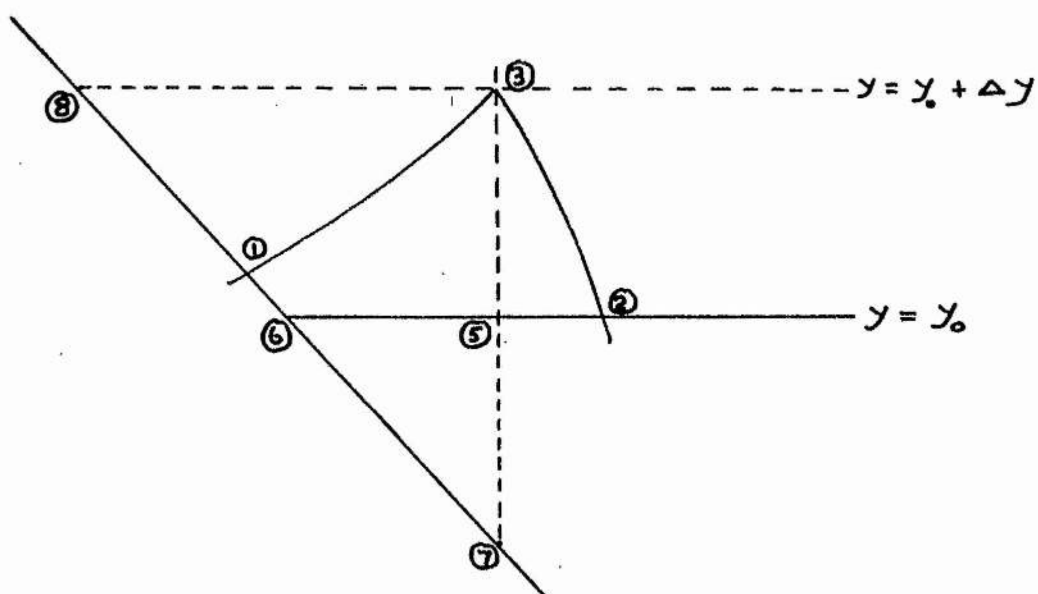


FIG. 23.

In this case we construct a shorter negative characteristic curve as shown. Now we indicate the changes that are made to our algorithm.

The second of equations (7.3.2) is replaced by

$$\phi_1^{(1)} = y_1^{(1)} = \left[\frac{\overline{C_-^{(0)}}\, y_3 - \phi_3}{\overline{C_-^{(0)}} - 1}\right] \quad ,$$

where $$\overline{C_-^{(0)}} = \tfrac{1}{2}\left[\left(\frac{d\phi}{dy}\right)_3^{(0)} + \left(\frac{d\phi}{dy}\right)_1^{(0)}\right] \quad ,$$

the notation being the same as that in (7.3.2).

In finding $u_1^{(1)}$ by interpolation, we now use points 7, 6, 8 and then $P_1^{(1)} = u_1^{(1)2}$, with $R_1 = 1$ a constant throughout the algorithm.

In equation (7.3.3), Δy is replaced by $(y_0 + \Delta y - y_1)$ and $D(y_0)$ is replaced by $D(y_1)$.

Now that we have established the method on a point to point basis we will explain the overall method. It can clearly be see seen from the third of equations (7.2.7) that, in the θ -y coordinate system near $y = 0$, the characteristic curves are almost parallel to the θ axis. In fact these curves coincide with the θ axis for $y = 0$ and it would be impossible to advance using $y = 0$ as a starting point for the integration in the θ -y system. In view of this we use the ϕ -y coordinate system for $0 \leq y \leq 1$, and then the θ -y system for $y > 1$. The change is made at $y = 1$ so that we can choose $\Delta\theta = \Delta\phi$.

7.4. Stability considerations.

If the method is stable then we can say that, given any numerical error at any time, its influence on the solution will decrease as we progress with the integration.

The usual stability condition, and the one used here, is the Courant-Friedrichs-Lewy condition and, with respect to the problem being discussed in this chapter, it can be stated mathematically as

$$\Delta y \leqslant \left| \overline{\left(\frac{dy}{d\tau}\right)} \right| \Delta \tau \, , \tag{7.4.1}$$

where τ again corresponds to either ϕ or θ and $\overline{\left(\frac{dy}{d\tau}\right)}$ is the average value of $\frac{dy}{d\tau}$ between the two end points of a characteristic curve.

It is evident from section 7.2. that the negative characteristic curve is the one on which $\frac{dy}{d\tau}$ is the smallest, and hence we use

$$\Delta y \leqslant \left| \overline{\left(\frac{dy}{d\tau}\right)_-} \right| \Delta \tau \, . \tag{7.4.2}$$

In constructing the numerical solution we do not know in advance how small Δy must be for global stability, and so we use a local stability condition at each mesh point. If the inequality (7.4.2) holds then point 1 , FIG. 22. , lies between points 5 and 6 and thus the check on stability can be made using this information.

To set an upper limit on Δy initially, near $y = 0$, we use the locally plane solution with $k = 0$. Thus we initially restrict Δy by

$$\frac{\Delta y}{\Delta \phi} \leqslant \frac{(\gamma-1)\left[1 - \sqrt{\frac{\gamma(\gamma-1)}{2}}\right]}{(\gamma+1)\sqrt{\frac{\gamma(\gamma-1)}{2}}} \, , \tag{7.4.3}$$

given $\Delta\phi$. In particular, for $\gamma = \frac{7}{5}, \frac{5}{3}$, this gives

$\frac{\Delta y}{\Delta\phi} \leq 0 \cdot 148$, $0 \cdot 085$ respectively and we then use

$\frac{\Delta y}{\Delta\phi} = \frac{1}{8}, \frac{1}{16}$ to give an initial value for Δy .

The stability condition is checked at each mesh point along the line $y = y_0 + \Delta y$ and if this condition is not met at any mesh point then we return to $y = y_0$ and replace Δy by $\frac{1}{2}\Delta y$. In the cases chosen, however, this was not found to be necessary and Δy was kept at a constant value up to $y = 1$ where we change over to the θ-y coordinate system. Still applying stability conditions we integrate up to $y = 2$ and then, since $d\theta$ is proportional to $y^{-1}dy$, we can double Δy. Indeed we double Δy again at $y = 4, 8$, etc. , until $\Delta y = \Delta\theta$. It was decided to keep Δy at this value for subsequent integration so that the truncation error, introduced by the application of the finite difference scheme, would be kept at the same order, i.e. $O(\Delta\theta^3)$.

7.5. Numerical results and comparison with the asymptotic analysis.

In order to compact the display of values of the gas variables we introduce a new variable, η, simply related to r and t by

$$\eta = \frac{r-1-t}{r_s-1-t}$$,where r_s is the shock radius at time t .

It is clear that $\eta = 0,1$ on the contact and shock fronts respectively. Also it can easily be shown that, as $t \to \infty$,

$$\eta \rightarrow \frac{\lambda-1}{\lambda_s-1} \;, \quad \delta = 1 \;,$$

$$\eta \rightarrow \frac{\lambda}{\lambda_s} \;, \quad \delta > 1 \;,$$

λ being the similarity variable of chapter 3 , and thus we may show the zeroth order inner solutions on the same graphs.

The numerical results were transferred from (θ, y), or (ϕ, y), to (r,t) coordinates by using the differential expressions connecting them and then interpolation was used to obtain information at the required value of t. In the subsequent displays the symbol ∞ is used to label information derived from the asymptotic analysis.

CASE I.

The values of the parameters in this case were chosen to be $\sigma = 2$, $\gamma = \frac{7}{5}$, $k = 1$. Clearly $k < (\sigma+1) < k_c$.

The variables u, P, R, as functions of η, are shown in FIGS. 24,25,26 respectively together with their corresponding asymptotic profiles and the functions $P\left(\frac{r}{r_s}\right)^{-k}$, $R\left(\frac{r}{r_s}\right)^{-k}$ are likewise plotted in FIGS. 27,28, the factor $\frac{r}{r_s}$ becoming $\frac{\lambda}{\lambda_s}$ for the asymptotic profiles. These last two mentioned functions are, apart from a time dependent scaling factor, actually observable in an experiment and thus it may seem that, for practical purposes at least, they are more interesting than their counterparts P and R.

Values of r_s at the times taken are given in TABLE 3 where we also include values of V_1 showing, once again, how quickly the flow settles down to its asymptotic state.

An inspection of FIG. 24. shows that the velocity, u, varies almost linearly with η, its profile steadily approaching that of the asymptotic solution.

There is nothing unexpected about FIG. 25. , for P, but, with regard to FIG. 26. , the value of R on the contact front first of all decreases as t increases from zero but then it increases, without bound eventually as P is reaching its asymptotic value. A perturbation solution for small y agrees with this occurrence for the parameters chosen here.

With regard to FIG. 27. , it can be seen that the profile of $P\left(\frac{r}{r_s}\right)^{-k}$ approaches its asymptotic form monotonically as t increases and, on considering FIG. 28. , the profile of $R\left(\frac{r}{r_s}\right)^{-k}$ does likewise and we notice that the value of this function on the contact front does not decrease initially.

Generally we can see that, in this case, the solutions for finite time do in fact approach the corresponding asymptotic solutions and, in the θ,y or ϕ_1,r coordinate systems, the difference is of the order of 0·05% which gives very good agreement (except for values of R in the immediate vicinity of the contact front where the inner solution is no longer valid), at about r = 8.

t	r_s	V_1
0	1	1·2
0·2	1·236	1·1655
0·6	1·694	1·1253
1·0	2·140	1·1079
1·5	2·691	1·0965
2·0	3·238	1·0910
6·0	7·580	1·0838
∞	∞	1·0832

TABLE 3.

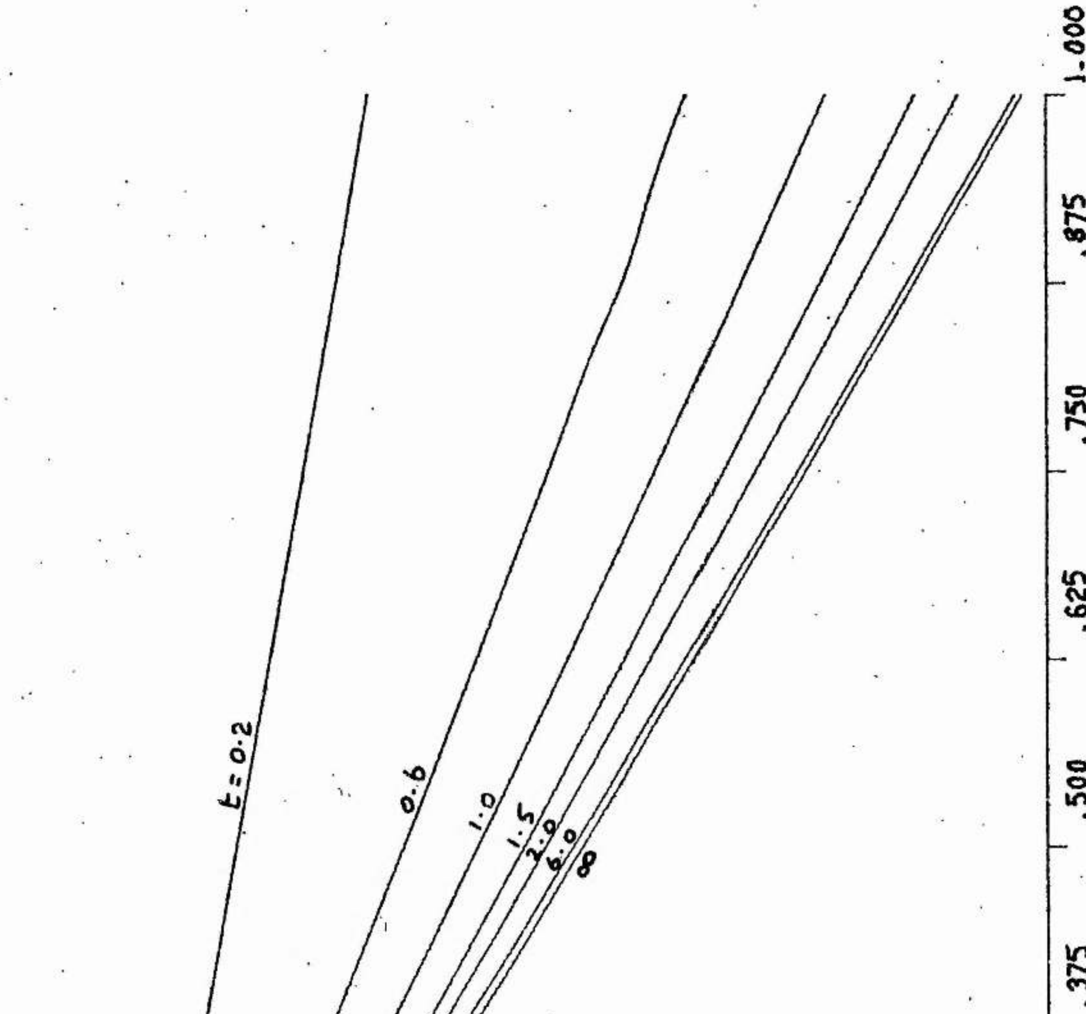
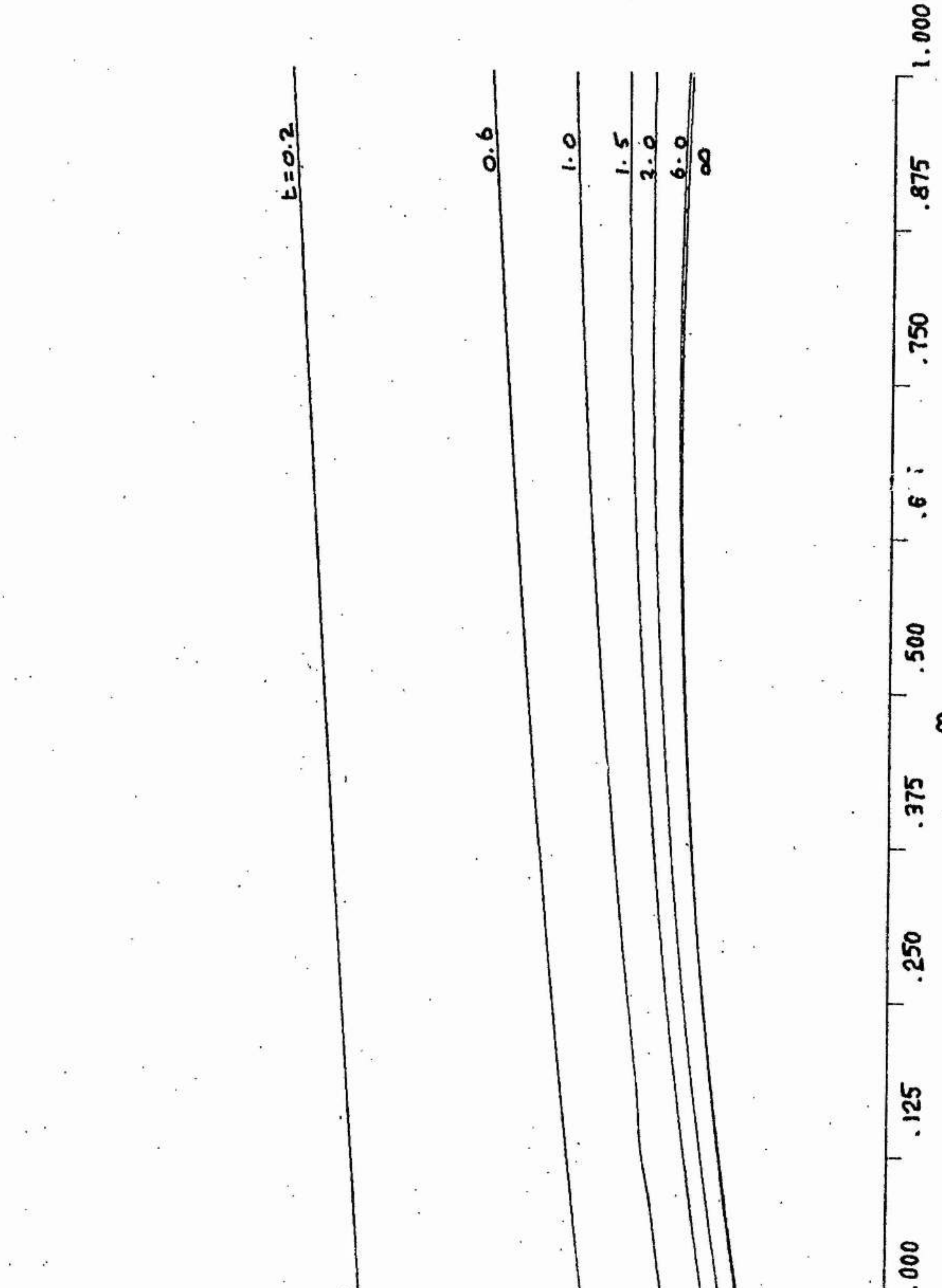
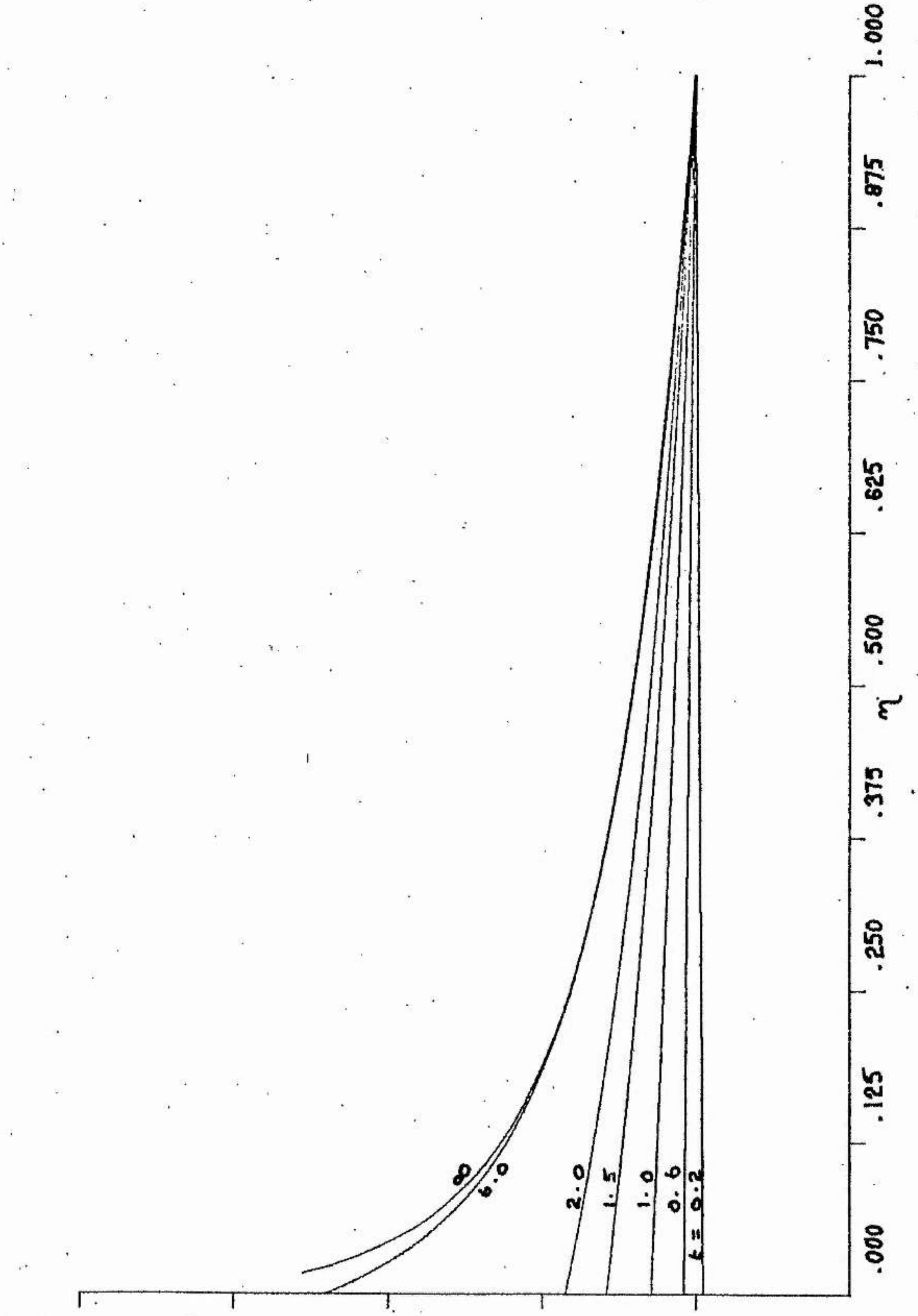

t = 0.2
0.6
1.0
1.5
2.0
6.0
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
1.000
.980
.960
.940
.920
.900

FIG. 24.



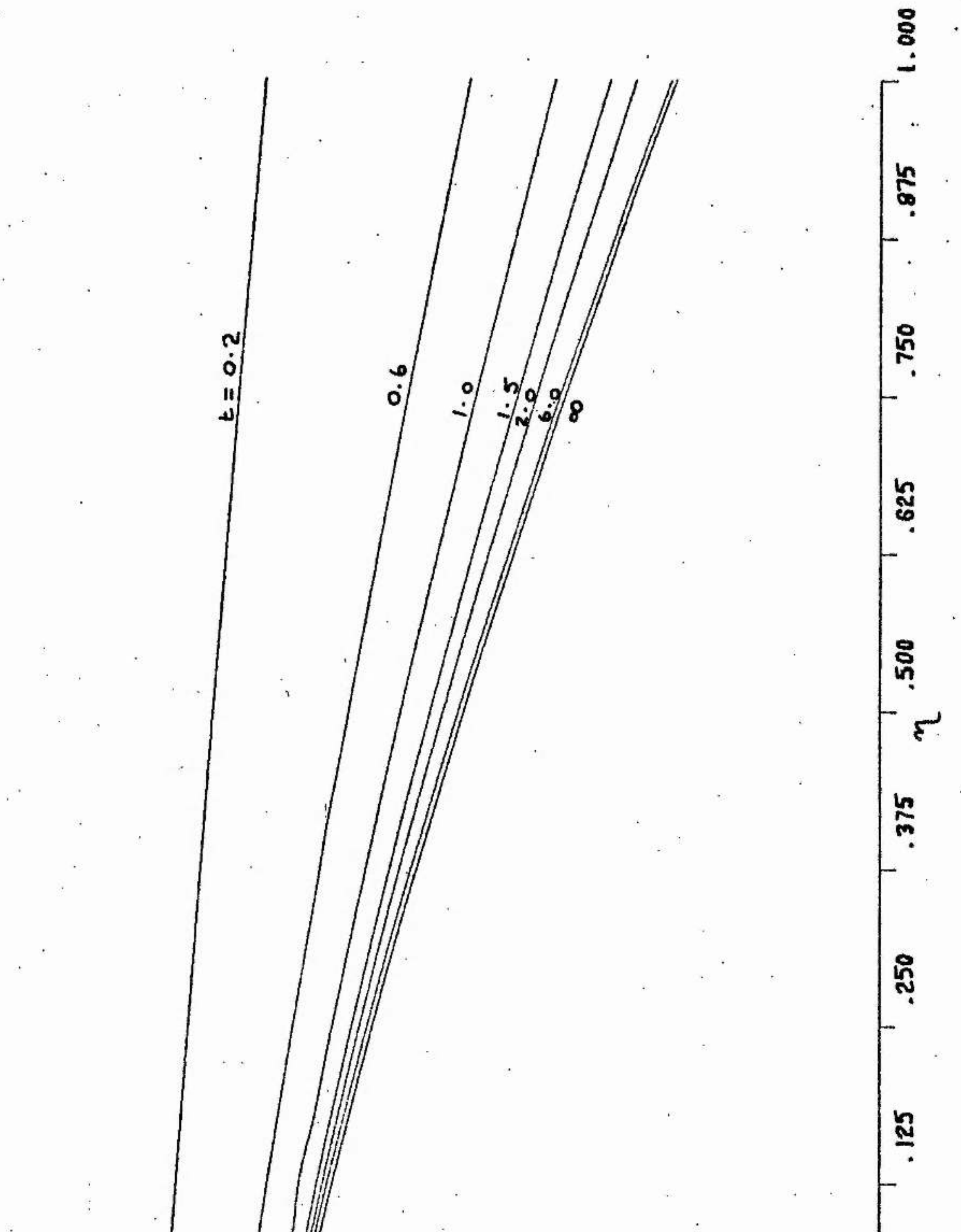
t=0.2
0.6
1.0
1.5
3.0
6.0
∞
P
η
.750
.800
.850
.900
.950
1.000
.000
.125
.250
.375
.500
.625
.750
.875
1.000

FIG. 25.



R
.800
1.000
1.200
1.400
1.600
1.800
η
.000
.125
.250
.375
.500
.625
.750
.875
1.000
∞
6.0
2.0
1.5
1.0
0.6
t = 0.2

FIG. 26.



t = 0.2
0.6
1.0
1.5
2.0
6.0
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
.750
.800
.850
.900
.950
1.000

FIG. 27.

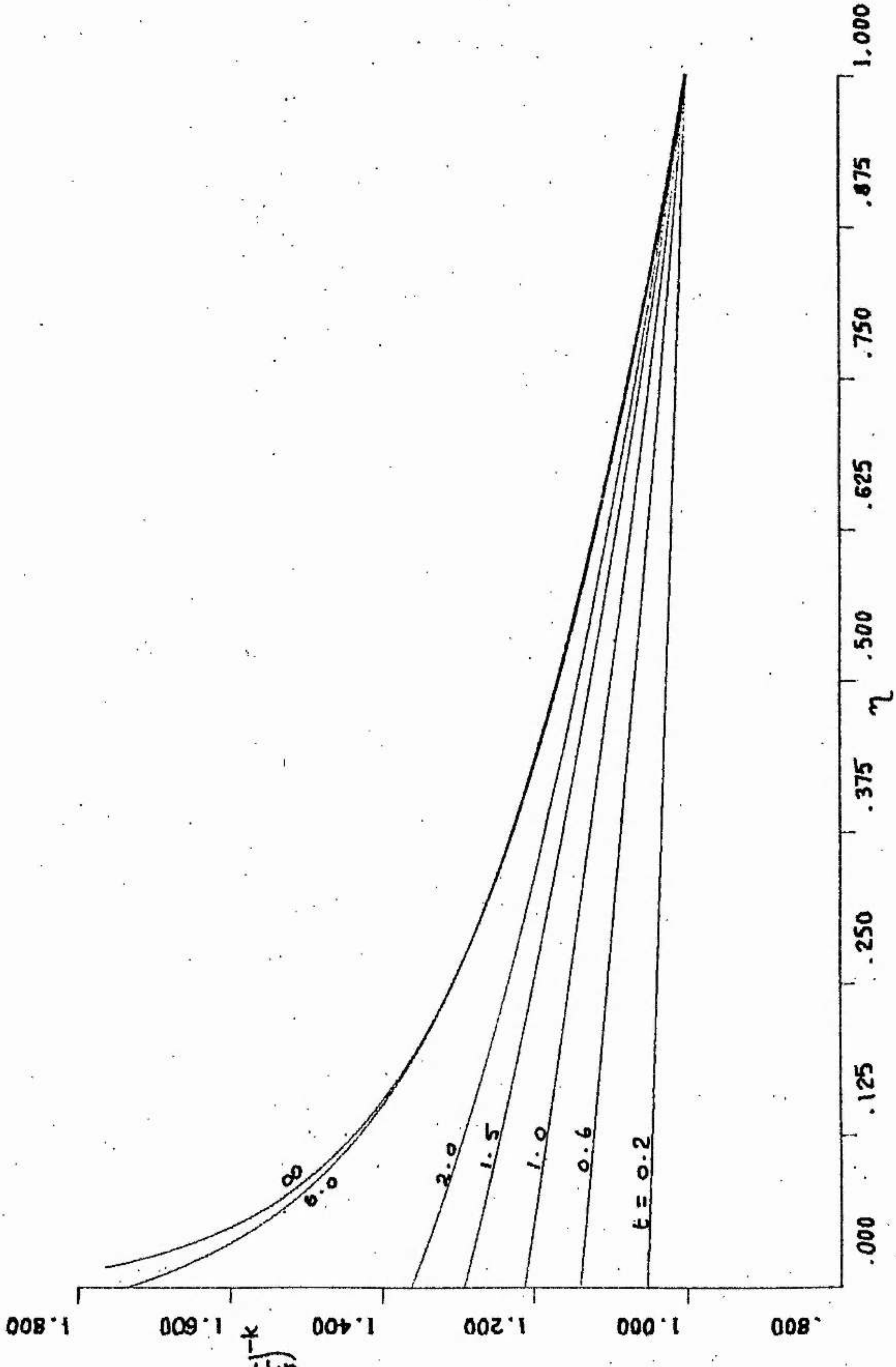

1.800
1.600
1.400
1.200
1.000
.800
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
t = 0.2
0.6
1.0
1.5
2.0
6.0
∞

FIG. 28.

CASE II.

In this case the parameter values are $\sigma = 2$, $\gamma = \frac{5}{3}$, $k = 3\cdot5$. Clearly $k > k_c > (\sigma+1)$.

Since the shock velocity increases without bound as t increases it was decided to display the functions u/u_s , P/P_s , where the subscript s refers to the shock. These functions are plotted in FIGS. 29,30 and R is shown in FIG. 31 . Also we display the physically more realistic functions $\frac{P}{P_s}\left(\frac{r}{r_s}\right)^{-k}$ and $R\left(\frac{r}{r_s}\right)^{-k}$, in FIGS. 32,33 . The respective asymptotic profiles are also shown. Values of r_s and V_1 are given in TABLE 4 so that u, for example, can be obtained if necessary.

It appears from these graphs that it takes a long time for the flow to settle down to its asymptotic state. The only explanation that can be given for this is that, in the asymptotic analysis, the dominant eigenvalue, w, may have a quite small real part or that b_1 could be quite large. It should be remembered that neither w nor b_1 can be determined from the asymptotic analysis alone.

The numerical solutions in this case are relied upon to support the asymptotic analysis of chapter 6. For the larger values of r curves of the form αr^β can be fitted by regressional analysis to P on the contact front and the shock velocity, V_1. For P on the contact front we obtain $\alpha = 184\cdot5$, $\beta = -1\cdot507$ and for V_1, $\alpha = 1\cdot505$, $\beta = 0\cdot0649$. The asymptotic analysis predicts $k - \gamma(\sigma+1)$ and ϵ respectively for the two indices, or $-1\cdot5$ and $0\cdot0647$ in this case. The very close correlation between these two values of β derived from the full numerical solutions and their respective values predicted from the asymptotic analysis strongly supports the forms of the outer and inner solutions of chapter 6.

Turning to FIGS. 29 to 33 we can see that, certainly for $\eta > 0{\cdot}5$, the profile of u/u_s has reached its predicted asymptotic form for the larger values of t, while the profile for P/P_s, at $t = 400$, closely follows its predicted asymptotic form for all η, although the percentage error becomes significant for $\eta < 0{\cdot}5$. The divergence of the finite time solutions, at large times, from the asymptotic solutions near the contact front should not be unexpected, however, since there the inner expansions of chapter 6 are no longer valid. The reason why there is a larger difference in the forms for u/u_s than in those for P/P_s is that, on the contact front, $u/u_s = O(t^{-0{\cdot}0647})$ while $P/P_s = (t^{-1{\cdot}5})$, so obviously P/P_s approaches zero quicker than does u/u_s for $t \to \infty$.

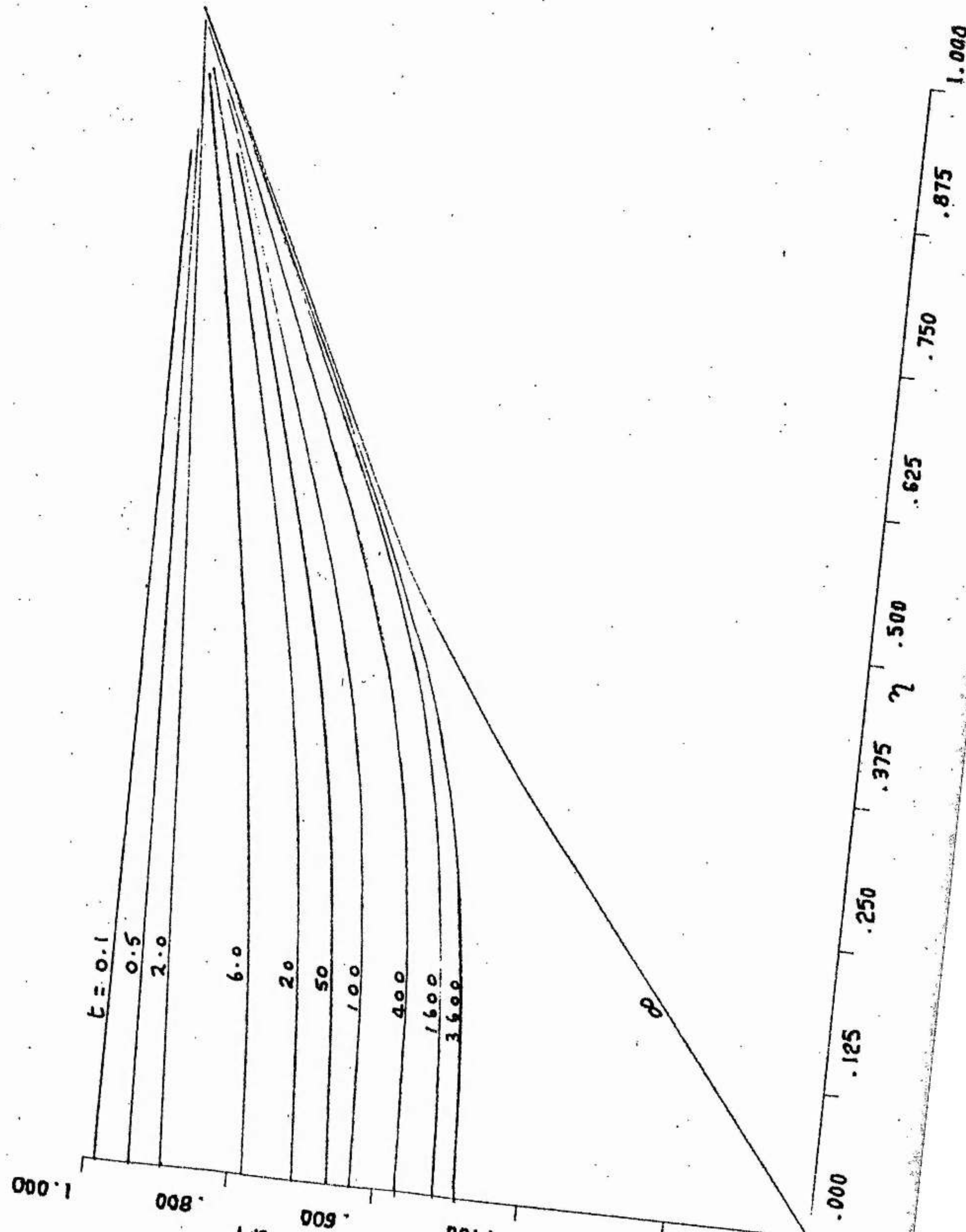
If we now take u/u_s as a function of θ, at constant r, then we can see from FIG. 34. that the two curves are quite close down to $\theta = 0{\cdot}15$ but then they diverge. This is again because of the breakdown of the inner expansions but here it shows that the outer region is compacted within a small range of θ near zero.

The conclusion we may draw from all of this is that there is overwhelming support for the asymptotic analysis of chapter 6. Even though the asymptotic analysis does not provide the complete outer solution it does predict the orders of magnitude of P and R which closely agree with those derived from the full numerical solutions.

Also the asymptotic analysis provides the value ϵ in $V_1 = \frac{(\gamma+1)}{2} b_0 r^{\epsilon} + \ldots\ldots$, which agrees, within an acceptable numerical error, with that provided by the full numerical solution which also gives b_0 . Although it takes a long time for the flow to settle down to its asymptotic state it can now be seen that it in fact does and we can now use the asymptotic analysis with complete confidence.

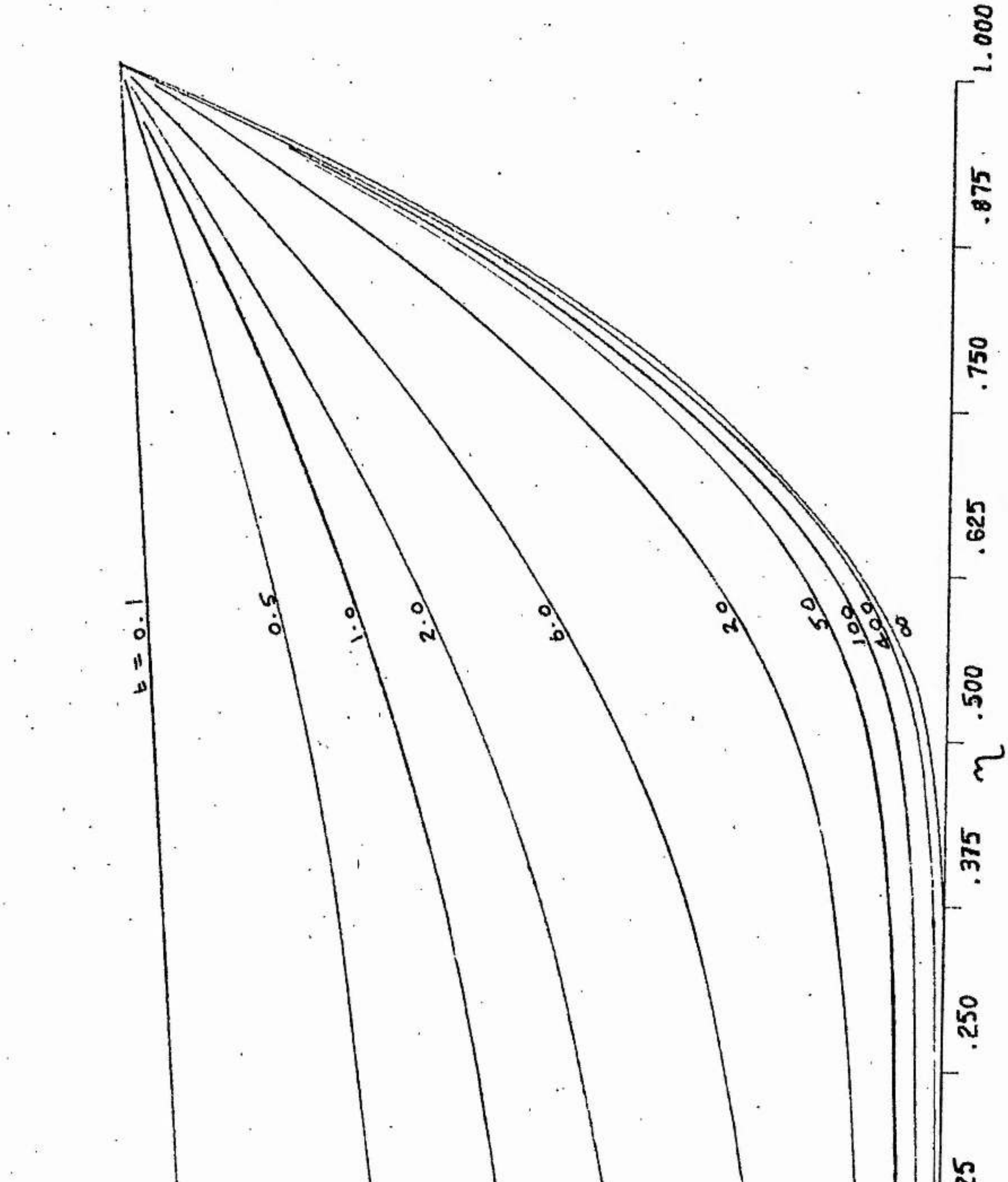
t	r_s	V_1
0	1	1·333
0·1	1·134	1·357
0·5	1·693	1·430
1·0	2·425	1·488
2·0	4·036	1·552
6·0	10·56	1·71
20	35·93	1·88
50	94·45	2·02
100	198·3	2·12
400	867·4	2·34
900	2046	2·47
1600	3749	2·57
2500	6000	2·65
3600	9116	2·72

TABLE 4.



t = 0.1
0.5
2.0
6.0
20
50
100
400
1600
3600
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
1.000
.800
.600
.400
.200
.000
u/u_s

FIG. 29.



t = 0.1
0.5
1.0
2.0
6.0
20
50
100
400
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
.000
.200
.400
.600
.800
1.000
p/p_s

FIG. 30.

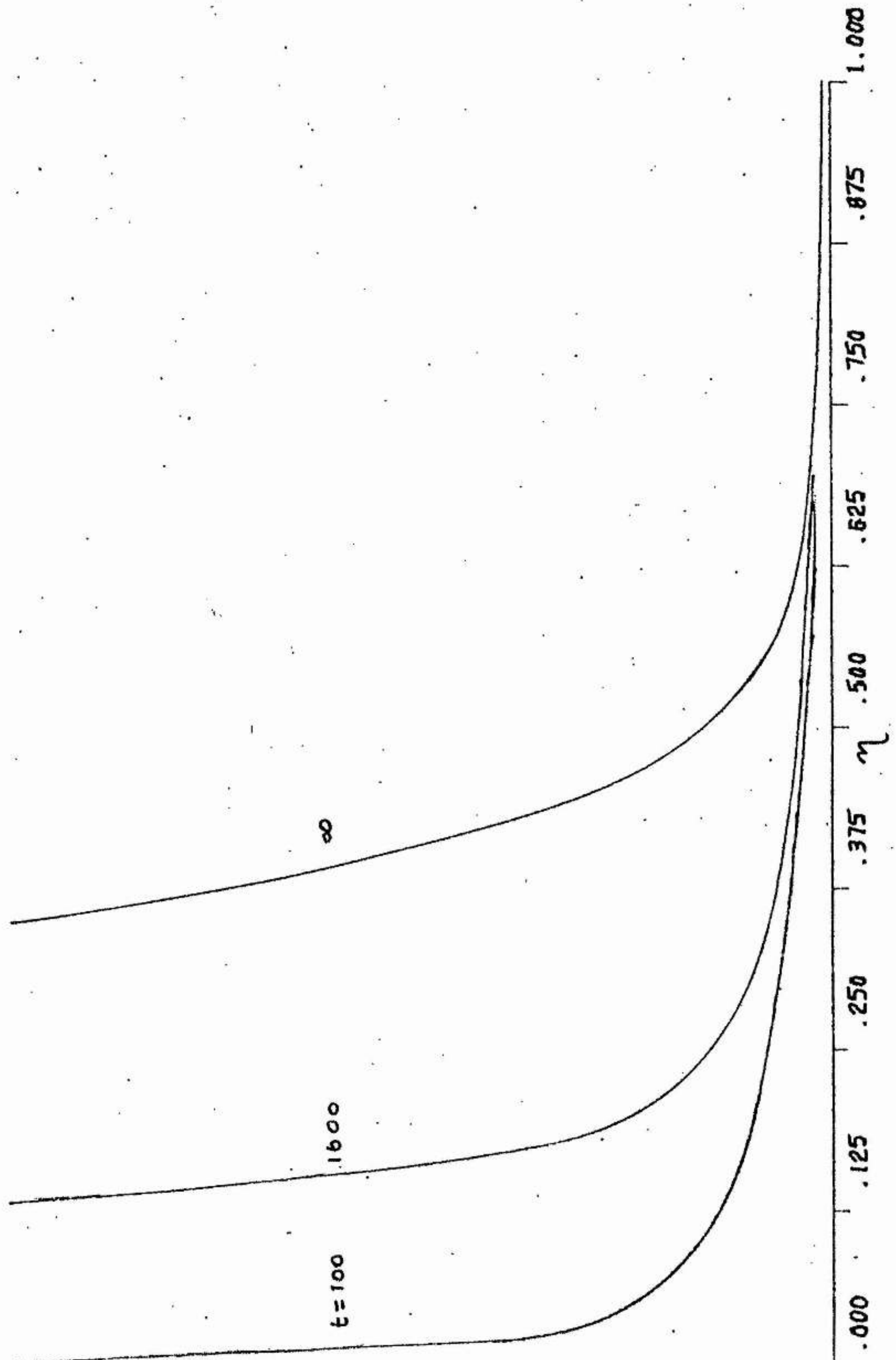

t=100
1600
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
.000
20.000
40.000
60.000
80.000
100.000
R

FIG. 31.

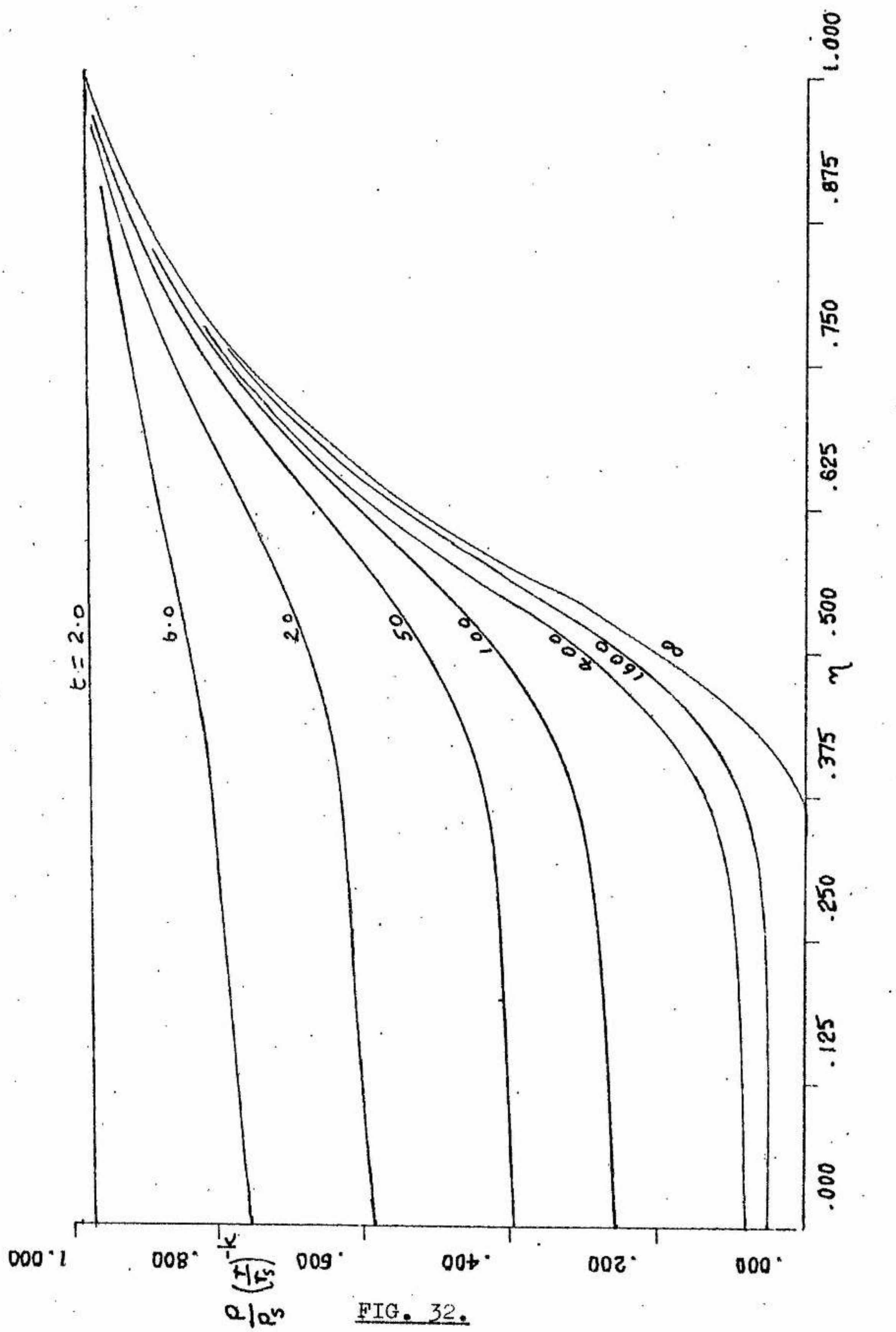
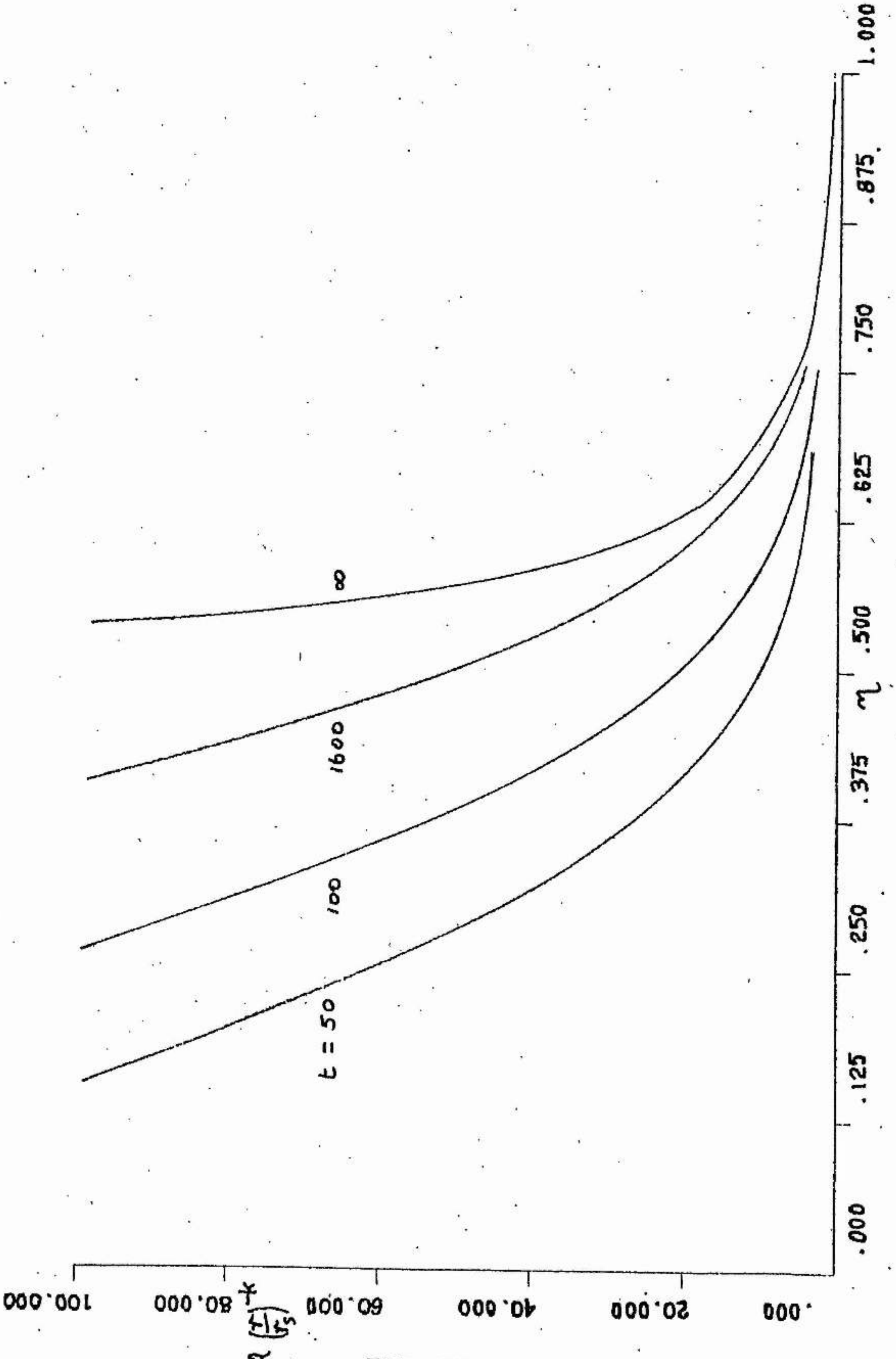

t = 2.0
6.0
20
50
100
200
1600
∞
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
.000
.200
.400
.600
.800
1.000
$\rho/\rho_s \left(\frac{T}{T_s}\right)^{-k}$

FIG. 32.



100.000
80.000
60.000
40.000
20.000
.000
.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
t = 50
100
1600
∞

FIG. 33.

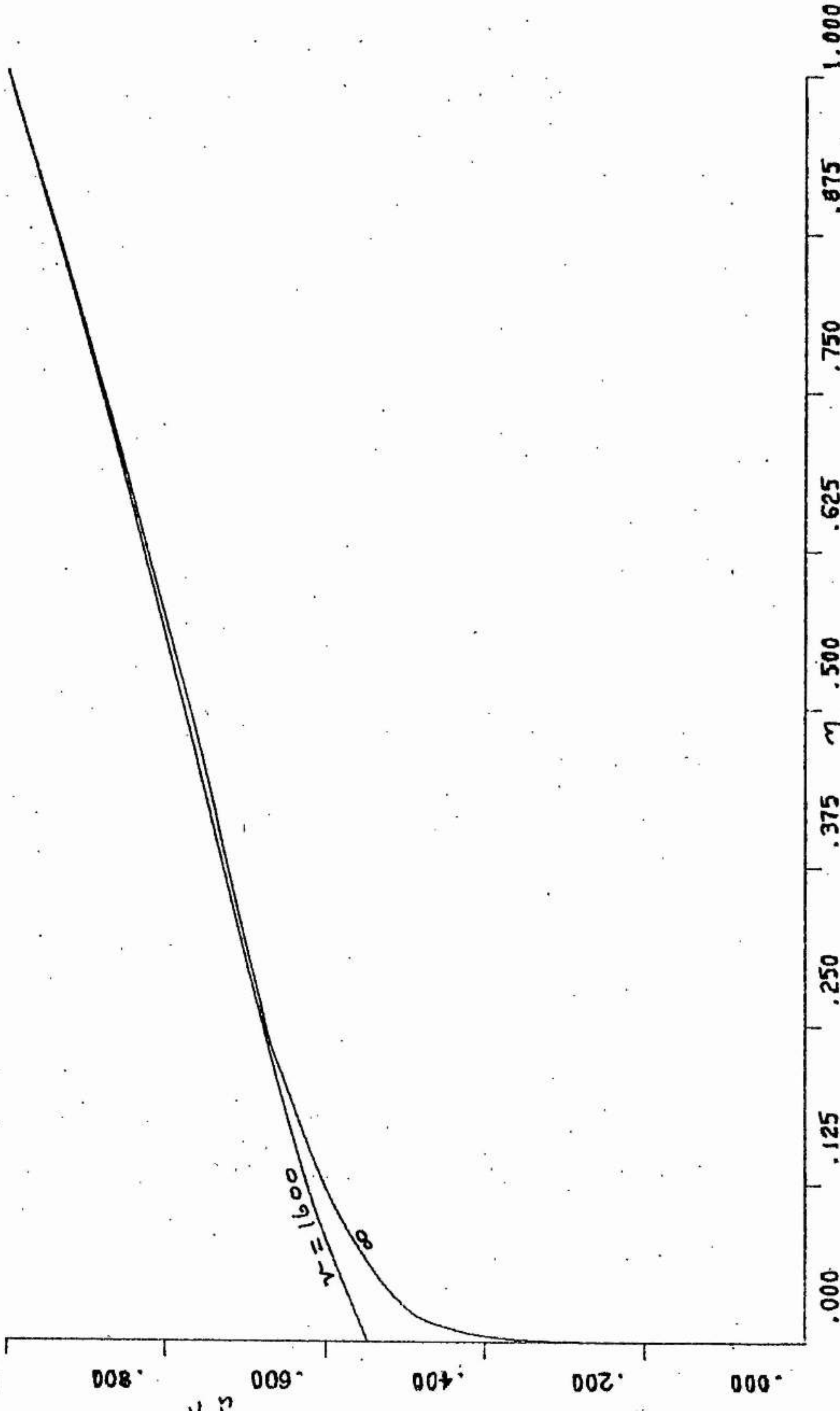

.000
.125
.250
.375
.500
.625
.750
.875
1.000
η
.000
.200
.400
.600
.800
1.000
ν = 1600
∞

FIG. 34.

Conclusions.

The unsteady continuum expansion of a gas into a near vacuum has been studied. We have examined the inviscid flow of the atmosphere between the contact front and the primary shock in the limit of the density and sound speed of the ambient atmosphere approaching zero. We have shown that, in this first approximation, the k-σ-parameter space is split up into two distinct regions where the asymptotic shock velocity is bounded and unbounded respectively, the dividing line, $k = k_c(\gamma,\sigma)$, being found in the first case. In the second case the extra parameter, δ, is shown to be a unique function of k, γ, σ.

In each case we have attempted to produce matched asymptotic expansions and it has been shown that the inner expansions, valid near the shock, breakdown as the contact front is approached. In the case $k < k_c$ we have obtained outer expansions and matched them with the inner expansions to zeroth and first orders. Matching to zeroth order has produced the constant asymptotic shock velocity and also the function k_c while matching to first order has provided an eigenvalue problem, a solution of which has been outlined. In the case $k > k_c$ outer expansions are sought but the equations do not form a closed set. Nevertheless it is shown that there exists a solution of the outer equations which matches with the inner expansions to zeroth and first orders. Zeroth order matching produces ϵ, where the shock velocity is, for large r, $V = b_0 r^{\epsilon} + \ldots\ldots$ and $\epsilon = 1 - \delta^{-1}$, but the coefficient b_0 cannot be determined by the asymptotic analysis alone. Matching to first order would yield an eigenvalue problem if more information on the outer solution were available.

Finally the full numerical solutions, which have not appeared before and which require a large amount of computer time to produce, have been used successfully to support and justify the use of the asymptotic analysis.

Appendix.

AI. Inner limit for the outer expansions, $k < k_c$.

$$u_1(\phi, r) = -\frac{(\sigma-k/\gamma)}{(\sigma+1-k)}\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} F_0(r)^{-1/\gamma}\left\{\left(\frac{\sigma+1-k}{\sigma+1-k/\gamma}\right) + \right.$$

$$\left. I^{*}\phi^{-\left(\frac{\sigma+1-k/\gamma}{\sigma+1-k}\right)} - \frac{2b_1(\sigma+1-k)}{\gamma\left[w-(\sigma+1-k/\gamma)\right]}\phi^{\frac{-w}{\sigma+1-k}} + \ldots\ldots \quad ,\right\}$$

$$\Pi_1(\phi, r) = \frac{\alpha_0(\gamma-1)F_0(r)^{-1/\gamma}}{(\sigma+1-k)(\gamma+1)}\left[kF_0(r) - r\frac{dF_0(r)}{dr}\right]\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}}$$

$$\left\{\left(\frac{\sigma+1-k}{\sigma+1-k/\gamma}\right) + I^{*}\phi^{-\left(\frac{\sigma+1-k/\gamma}{\sigma+1-k}\right)} - \frac{2b_1(\sigma+1-k)}{\gamma\left[w-(\sigma+1-k/\gamma)\right]}\phi^{\frac{-w}{\sigma+1-k}} + \ldots\right\}$$

$$+\alpha_1(r) \quad ,$$

where $\Pi_1(1,r) = \alpha_1(r) = O(1)$, as $r \to \infty$,

$$R_0(\phi, r) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}}\phi^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{1 - \frac{2b_1}{\gamma}\phi^{\frac{-w}{\sigma+1-k}} + \ldots\ldots\right\}F_0(r)^{1/\gamma},$$

$$R_1(\phi, r) = \frac{(\gamma-1)\left[kF_0(r) - r\frac{dF_0(r)}{dr}\right]}{\gamma(\gamma+1)(\sigma+1-k)\,F_0(r)}\phi\left\{\left(\frac{\sigma+1-k}{\sigma+1-k/\gamma}\right) + \right.$$

$$I^{*}\phi^{-\left(\frac{\sigma+1-k/\gamma}{\sigma+1-k}\right)} - \frac{2b_1}{\gamma}\,(\sigma+1-k)\left[\frac{1}{(\sigma+1-k/\gamma)} + \frac{1}{w-(\sigma+1-k/\gamma)}\right]\phi^{\frac{-w}{\sigma+1-k}}$$

$$+ \ldots\ldots\Big\} + \alpha_1(r)\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}}\phi^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\,F_0(r)^{-1+1/\gamma}\left\{1 - \frac{2b_1}{\gamma}\phi^{\frac{-w}{\sigma+1-k}} + \ldots\right\},$$

$$\text{where}\quad I^{*} = \int_1^{\infty}\phi^{\frac{k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{\left[\frac{V\left(\phi^{\frac{1}{\sigma+1-k}}\right)}{b_0}\right]^{\frac{2}{\gamma}} - 1\right\}d\phi - \left[\frac{\sigma+1-k}{\sigma+1-k/\gamma}\right],$$

for $k < (\sigma+1)$,

$$u_1(\psi,r) = \frac{-(\sigma-k/\gamma)(\gamma-1)}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} e^{k(\gamma-1)\psi/\gamma}F_0(r)\left\{\frac{\gamma}{k(\gamma-1)} +\right.$$

$$\left. I^{*}e^{-k(\gamma-1)\psi/\gamma} - \frac{2b_1\,e^{-w\psi}}{\gamma[w-k(\gamma-1)/\gamma]} + \ldots\ldots\right\},$$

$$\Pi_1(\psi,r) = \frac{-2k\,(\gamma-1)^2\,(\sigma-k/\gamma)}{\gamma\,(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} e^{k(\gamma-1)\psi/\gamma}F_0(r)^{-1/\gamma}\left\{\frac{\gamma^2}{k^2(\gamma-1)^2}\right.$$

$$+ I^{*}\psi e^{-k(\gamma-1)\psi/\gamma} - J^{*}e^{-k(\gamma-1)\psi/\gamma} + \frac{2b_1\,e^{-w\psi}}{\gamma\left[w-k(\gamma-1)/\gamma\right]} + \ldots\ldots$$

$$+ \alpha_0\,\frac{(\gamma-1)}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} e^{k(\gamma-1)\psi/\gamma}F_0(r)^{-1/\gamma}\left[kF_0(r) - r\frac{dF_0(r)}{dr}\right]\left\{\frac{\gamma}{k(\gamma-1)}\right.$$

$$+ I^{*} e^{-k(\gamma-1)\psi/\gamma} - \frac{2b_1 \, e^{-w\psi}}{\gamma\left[w-k(\gamma-1)/\gamma\right]} + \ldots\ldots \Bigg\} \quad + \alpha_1(r) \quad ,$$

where $\Pi_1(0,r) = \alpha_1(r) = O(1)$, as $r \to \infty$,

$$R_0(\psi, r) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} e^{-k(\gamma-1)\psi/\gamma}\left\{1 - \frac{2b_1}{\gamma} e^{-w\psi} + \ldots\ldots\right\} \quad ,$$

$$R_1(\psi, r) = -\frac{k(\sigma-k/\gamma)(\gamma-1)^2}{\gamma^2\alpha_0 \; (\gamma+1)^2}\left\{\frac{\gamma^2}{k^2(\gamma-1)^2} + I^{*}\psi e^{-k(\gamma-1)\psi/\gamma} - \right.$$

$$\left. J^{*} e^{-k(\gamma-1)\psi/\gamma} + \frac{2b_1}{\gamma}\left[\frac{1}{\{w-k(\gamma-1)/\gamma\}^2} - \frac{\gamma^2}{k^2(\gamma-1)^2}\right] e^{-w\psi} + \ldots\right\}$$

$$+ \alpha_0 \; \frac{(\gamma-1)}{(\gamma+1)}\left[kF_0(r) - r\frac{dF_0(r)}{dr}\right]\left\{\frac{\gamma}{k(\gamma-1)} + I^{*} e^{-k(\gamma-1)\psi/\gamma} - \right.$$

$$\left. \frac{2b_1}{\gamma}\left[\frac{1}{w-k(\gamma-1)/\gamma} + \frac{\gamma}{k(\gamma-1)}\right] e^{-w\psi} + \ldots\ldots\right\}$$

$$+ \frac{\alpha_1(r)}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} F_0(r)^{-1+1/\gamma} \; e^{-k(\gamma-1)\psi/\gamma} \; ,$$

$$\text{where } I^{*} = \int_0^{\infty} e^{k(\gamma-1)\psi/\gamma}\left\{\left[\frac{V(e^{\psi})}{b_0}\right]^{\frac{2}{\gamma}} - 1\right\} d\psi - \frac{\gamma}{k(\gamma-1)} \quad ,$$

$$J^{*} = \int_0^\infty \psi e^{k(\gamma-1)\psi/\gamma} \left\{ \left[\frac{V(e^{\psi})}{b_0}\right]^{\frac{2}{\gamma}} - 1 \right\} d\psi + \frac{\gamma^2}{k^2(\gamma-1)^2} \quad ,$$

for $k = (\sigma+1)$,

$$u_1(\phi, r) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(k-\sigma-1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} F_0^{-1/\gamma}(r)\, \phi^{-\left(\frac{\sigma+1-k/\gamma}{k-\sigma-1}\right)} \left\{ \left(\frac{k-\sigma-1}{\sigma+1-k/\gamma}\right) + \right.$$

$$\left. I^{*}\phi^{\frac{\sigma+1-k/\gamma}{k-\sigma-1}} - \frac{2b_1(k-\sigma-1)}{\gamma\left[w-(\sigma+1-k/\gamma)\right]}\phi^{\frac{w}{k-\sigma-1}} + \ldots\ldots \right\} \quad ,$$

$$\Pi_1(\phi, r) = -\frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{(k-\sigma-1)^2\ (\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} F_0^{-1/\gamma}(r)\, \phi^{\frac{k(\gamma+1)/\gamma-2(\sigma+1)}{(k-\sigma-1)}}$$

$$\left\{ \frac{(k-\sigma-1)^2}{(\sigma+1-k/\gamma)\left[2(\sigma+1)-k(\gamma+1)/\gamma\right]} + J^{*}\phi^{\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}} - \right.$$

$$\left. I^{*}\phi^{\frac{\sigma+1-k/\gamma}{k-\sigma-1}} + \frac{2b_1(k-\sigma-1)^2\ \phi^{\frac{w}{k-\sigma-1}}}{\gamma\left[w-(\sigma+1-k/\gamma)\right]\left[w-2(\sigma+1)+k(\gamma+1)/\gamma\right]} + \ldots\ldots \right\}$$

$$+ \alpha_1(r)$$

where $\Pi_1(1, r) = \alpha_1(r) = O(1)$, as $r \to \infty$,

$$R_0(\phi, r) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} \phi^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\, F_0^{1/\gamma}(r) \left\{ 1 - \frac{2b_1}{\gamma}\phi^{\frac{w}{k-\sigma-1}} + \ldots\ldots \right\} \quad ,$$

$$R_1(\phi,r) = -\frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{\gamma\alpha_0\ (k-\sigma-1)^2\ (\gamma+1)^2\ F_0(r)}\ \phi^2$$

$$\left\{\frac{(k-\sigma-1)^2}{(\sigma+1-k/\gamma)\left[2(\sigma+1)-k(\gamma+1)/\gamma\right]} + J^* \phi^{\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}} -\right.$$

$$I^*\phi^{\frac{\sigma+1-k/\gamma}{k-\sigma-1}} + \frac{2b_1}{\gamma}\left[\frac{(k-\sigma-1)^2}{\{w-(\sigma+1-k/\gamma)\}\{w-2(\sigma+1)+k(\gamma+1)/\gamma\}} - 1\right]\phi^{\frac{w}{k-\sigma-1}}$$

$$\left.+\ \ldots\right\} + \frac{\alpha_1(r)}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}}\phi^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\ F_0(r)^{-1+1/\gamma} + \ldots\ldots ,$$

$$\text{where } I^* = \int_0^1 \phi^{\frac{-k(\gamma-1)}{\gamma(k-\sigma-1)}}\left\{\left[\frac{V(\phi^{\frac{-1}{k-\sigma-1}})}{b_0}\right]^{\frac{2}{\gamma}} - 1\right\}d\phi - \frac{k-\sigma-1}{\sigma+1-k/\gamma} ,$$

$$J^* = \int_0^1 \phi^{-\frac{\sigma+1-k/\gamma}{k-\sigma-1}}\left\{\left[\frac{V(\phi^{\frac{-1}{k-\sigma-1}})}{b_0}\right]^{\frac{2}{\gamma}} - 1\right\}d\phi - \frac{k-\sigma-1}{2(\sigma+1)-k(\gamma+1)/\gamma} ,$$

for $k > (\sigma+1)$.

The special case of $k = 0$ yields

$$u_1(\phi,r) = -\frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} F_0(r)^{-1/\gamma}\ \left\{1 + \frac{2b_1(\sigma+1)}{\gamma(\sigma+1-w)}\phi^{\frac{-w}{(\sigma+1)}} +\right.$$

$$\left. I^*\phi^{-1} + \ldots\ldots\right\} ,$$

$$\Pi_2(\phi,r) = \alpha_2(r) - \frac{2\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} F_0(r)^{-1/\gamma}\ \phi^2\left\{\frac{1}{2} + \right.$$

$$\left.\frac{2b_1\ (\sigma+1)^2}{\gamma(\sigma+1-w)\left[2(\sigma+1)-w\right]}\phi^{\frac{-w}{\sigma+1}} + I^{*}\phi^{-1} + \ldots\ldots\right\} ,$$

$$R_0(\phi,r) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} F_0(r)^{1/\gamma}\left\{1 + \frac{2b_1}{\gamma}\phi^{\frac{-w}{\sigma+1}} + \ldots\ldots\right\} ,$$

$$R_2(\phi,r) = -\frac{2\sigma(\gamma-1)\ F_0(r)^{-1}}{\gamma(\sigma+1)(\gamma+1)^2\alpha_0}\ \phi^2\left\{\frac{1}{2} + \right.$$

$$\left.\frac{2b_1}{\gamma}\left[\frac{(\sigma+1)^2}{(\sigma+1-w)\left\{2(\sigma+1)-w\right\}} + \frac{1}{2}\right]\phi^{\frac{-w}{\sigma+1}} + I^{*}\phi^{-1} + \ldots\ldots\right\} + ,$$

$$\frac{\alpha_2(r)}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} F_0(r)^{-1+1/\gamma}\left\{1 + \frac{2b_1}{\gamma}\phi^{\frac{-w}{\sigma+1}} + \ldots\ldots\right\} ,$$

where $$I^{*} = \int_1^{\infty}\left\{\left[\frac{V(\phi^{\frac{1}{\sigma+1}})}{b_0}\right]^{\frac{2}{\gamma}} - 1\right\} d\phi - 1 .$$

AII. Matching conditions for the inner expansions, $k < k_c$.

$$U_0(\phi_1) = 1 - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} + \ldots\ldots ,$$

$$U_0^*(\phi_1, r) = - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} G_0(r) + \ldots\ldots ,$$

$$U_1(\phi_1) = - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* + \ldots\ldots ,$$

$$U_1^*(\phi_1, r) = - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* G_0(r) + \ldots\ldots ,$$

$$U_2(\phi_1) = \frac{2b_1 \ (\sigma-k/\gamma)(\gamma-1)}{\gamma(\gamma+1)\left[w-(\sigma+1-k/\gamma)\right]} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{-\left[\frac{w-(\sigma+1-k/\gamma)}{\sigma+1-k}\right]} + \ldots. ,$$

$$P_0(\phi_1) = \alpha_0 + \frac{k\alpha_0 \ (\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right) \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} + \ldots\ldots ,$$

$$P_0^*(\phi_1, r) = \alpha_0 H_0(r) - \frac{\alpha_0(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} H_2(r) + \ldots ,$$

$$P_1(\phi_1) = \alpha_1(\infty) + \frac{k\alpha_0 \ (\gamma-1)}{(\sigma+1-k)(\gamma+1)} I^* \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} + \ldots\ldots ,$$

$$P_1^*(\phi_1, r) = \left[\alpha_1(r) - \alpha_1(\infty)\right] - \frac{\alpha_0 \ (\gamma-1)}{(\sigma+1-k)(\gamma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* H_2(r) + \ldots.. ,$$

$$P_2(\phi_1) = -\frac{2b_1\ k\,\alpha_0\ (\gamma-1)}{\gamma(\gamma+1)\left[w-(\sigma+1-k/\gamma)\right]}\ \phi_1^{-\left[\frac{w-(\sigma+1-k/\gamma)}{\sigma+1-k}\right]} + \ldots\ldots ,$$

$$\rho_0(\phi_1) = \phi_1^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} + \frac{k(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k/\gamma)}\ \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} + \ldots\right\},$$

$$\rho_0^*(\phi_1,r) = \phi_1^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} G_1(r) - \frac{(\gamma-1)\ H_4(r)}{\gamma(\gamma+1)(\sigma+1-k/\gamma)}\ \phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} + \ldots\right\},$$

$$\rho_1(\phi_1) = \phi_1^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{\alpha_1(\infty)\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} + \frac{k(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k)}\ I^* + \ldots\right\},$$

$$\rho_1^*(\phi_1,r) = \phi_1^{\frac{-k(\gamma-1)}{\gamma(\sigma+1-k)}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} G_2(r) - \frac{(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-k)}\ I^* H_3(r) + \ldots\ldots\right\},$$

$$\rho_2(\phi_1) = -\frac{2b_1}{\gamma}\ \phi_1^{-\left[\frac{w+k(\gamma-1)/\gamma}{\sigma+1-k}\right]}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} + \frac{k(\gamma-1)}{\gamma(\gamma+1)}\left[\frac{1}{\sigma+1-k/\gamma} + \frac{1}{w-(\sigma+1-k/\gamma)}\right]\phi_1^{\frac{\sigma+1-k/\gamma}{\sigma+1-k}} + \ldots\ldots\right\},$$

for $0 < k < (\sigma+1)$.

$$U_0(\phi_1) = 1 - \frac{(\sigma-k/\gamma)\gamma}{k(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{k(\gamma-1)/\gamma} + \ldots\ldots ,$$

$$U_0^*(\phi_1,r) = -\frac{(\sigma-k/\gamma)\gamma}{k(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{k(\gamma-1)/\gamma}\ G_0(r) + \ldots\ldots ,$$

$$U_1(\phi_1) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* + \ldots\ldots ,$$

$$U_1^*(\phi_1, r) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* G_0(r) + \ldots\ldots ,$$

$$U_2(\phi_1) = \frac{2b_1(\sigma-k/\gamma)(\gamma-1)}{\gamma(\gamma+1)\left[w-k(\gamma-1)/\gamma\right]}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{-w+k(\gamma-1)/\gamma} + \ldots\ldots ,$$

$$P_0(\phi_1) = \alpha_0 - \gamma\left[\frac{2(\sigma-k/\gamma)}{k(\gamma+1)^2} - \frac{\alpha_0}{\gamma+1}\right]\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{k(\gamma-1)/\gamma} + \ldots. ,$$

$$P_0^*(\phi_1, r) = \alpha_0 H_0(r) - \frac{2(\sigma-k/\gamma)}{k\gamma(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{k(\gamma-1)/\gamma} G_0(r) -$$

$$\frac{\gamma\alpha_0}{k(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^{k(\gamma-1)/\gamma} H_2'(r) -$$

$$\frac{2k(\gamma-1)^2(\sigma-k/\gamma)}{\gamma(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* F_0^{-1/\gamma}(r) \ln r \; r^{-k(\gamma-1)/\gamma} + \ldots\ldots ,$$

$$P_1(\phi_1) = -\frac{2k(\gamma-1)^2(\sigma-k/\gamma)}{\gamma(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\left\{I^* \ln \phi_1 - J^*\right\} +$$

$$k\alpha_0\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* + \alpha_1(\infty) + \ldots\ldots ,$$

$$P_1^*(\phi_1, r) = -\frac{2k(\gamma-1)^2(\sigma-k/\gamma)}{\gamma(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\left\{I^* \ln \phi_1 - J^*\right\} G_0(r) -$$

$$\alpha_0\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^{*}H_2(r) + \left[\alpha_1(r) - \alpha_1(\infty)\right] + \ldots\ldots \quad ,$$

$$P_2(\phi_1) = -\frac{2b_1\ \phi_1^{-w+k(\gamma-1)/\gamma}}{\gamma\left[w-k(\gamma-1)/\gamma\right]^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\left\{\frac{2k(\gamma-1)^2(\sigma-k/\gamma)}{\gamma(\gamma+1)^2} +\right.$$

$$\left. k\alpha_0\left(\frac{\gamma-1}{\gamma+1}\right)\left[w-k(\gamma-1)/\gamma\right]\right\} + \ldots\ldots \quad ,$$

$$\rho_0(\phi_1) = \phi_1^{-k(\gamma-1)/\gamma}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} + \left[\frac{1}{\gamma+1} - \frac{2(\sigma-k/\gamma)}{\alpha_0 k(\gamma+1)^2}\right]\phi_1^{k(\gamma-1)/\gamma} + \ldots\right\} ,$$

$$\rho_0^{*}(\phi_1, r) = -\frac{2k(\sigma-k/\gamma)(\gamma-1)^2}{\gamma^2\alpha_0\ (\gamma+1)^2}\ I^{*}\ \phi_1^{-k(\gamma-1)/\gamma}(\ln r)\, r^{-k(\gamma-1)/\gamma} -$$

$$\frac{\alpha_0\gamma H_3(r)}{k(\gamma+1)} + \ldots\ldots \quad ,$$

$$\rho_1(\phi_1) = -\frac{k(\sigma-k/\gamma)(\gamma-1)^2}{\gamma^2\alpha_0(\gamma+1)^2}\left\{I^{*}\ln\phi_1 - J^{*}\right\}\phi_1^{-k(\gamma-1)/\gamma} +$$

$$k\alpha_0\left(\frac{\gamma-1}{\gamma+1}\right) I^{*}\phi_1^{-k(\gamma-1)/\gamma} + \frac{\alpha_1(\infty)}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}}\phi_1^{-k(\gamma-1)/\gamma} + \ldots\ldots \quad ,$$

$$\rho_1^{*}(\phi_1, r) = -\alpha_0\left(\frac{\gamma-1}{\gamma+1}\right) I^{*}\phi_1^{-k(\gamma-1)/\gamma}\ H_3(r) +$$

$$\frac{1}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}}\phi_1^{-k(\gamma-1)/\gamma}\ G_2(r) + \ldots\ldots \quad ,$$

$$\rho_2(\phi_1) = -\frac{2b_1}{\gamma}\phi_1^{-w-k(\gamma-1)/\gamma}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} +\right.$$

$$\frac{k(\sigma-k/\gamma)(\gamma-1)^2}{\gamma^2\alpha_0\ (\gamma+1)^2}\left[\frac{1}{[w-k(\gamma-1)/\gamma]^2}-\frac{\gamma^2}{k^2(\gamma-1)^2}\right]\phi_1^{k(\gamma-1)/\gamma} -$$

$$\left.\frac{k^2(\sigma-k/\gamma)(\gamma-1)^2}{\gamma^2\ (\gamma+1)^2}\left[\frac{1}{w-k(\gamma-1)/\gamma}+\frac{\gamma}{k(\gamma-1)}\right]\phi_1^{k(\gamma-1)/\gamma} + \ldots\ldots\right\} ,$$

for $k = (\sigma+1)$.

$$U_0(\phi_1) = 1 - \frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{-\left(\frac{\sigma+1-k/\gamma}{k-\sigma-1}\right)} + \ldots\ldots ,$$

$$U_0^*(\phi_1,r) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(\sigma+1-k/\gamma)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{-\left(\frac{\sigma+1-k/\gamma}{k-\sigma-1}\right)} G_0(r) + \ldots ,$$

$$U_1(\phi_1) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(k-\sigma-1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* + \ldots\ldots ,$$

$$U_1^*(\phi_1,r) = -\frac{(\sigma-k/\gamma)(\gamma-1)}{(k-\sigma-1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* G_0(r) + \ldots\ldots ,$$

$$U_2(\phi_1) = \frac{2b_1(\sigma-k/\gamma)(\gamma-1)}{\gamma[w-(\sigma+1-k/\gamma)]\ (\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{\frac{w-(\sigma+1-k/\gamma)}{k-\sigma-1}} + \ldots\ldots ,$$

$$P_0(\phi_1) = \alpha_0 - \frac{2(\sigma-k/\gamma)(\gamma-1)}{[2(\sigma+1)-k(\gamma+1)/\gamma](\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{-\left[\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}\right]} + \ldots ,$$

$$P_0^*(\phi_1,r) = \alpha_0 H_0(r) - \frac{2(\sigma-k/\gamma)(\gamma-1)\ \phi_1^{-\left[\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}\right]}}{[2(\sigma+1)-k(\gamma+1)/\gamma]\ (\gamma+1)^2} G_0(r)\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} -$$

$$\frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)\ J^{*}\ F_0(r)^{1/\gamma}}{(k-\sigma-1)^2\ (\gamma+1)^2\ r^{2(\sigma+1)-k(\gamma+1)/\gamma}} + \frac{\alpha_1(r)}{r^{2(\sigma+1)-k(\gamma+1)/\gamma}} + \ldots$$

$$P_1(\phi_1) = \frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{(k-\sigma-1)^2(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^{*} + \ldots\ldots ,$$

$$P_1^{*}(\phi_1, r) = \frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{(k-\sigma-1)^2(\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^{*}\ G_0(r) + \ldots\ldots ,$$

$$P_2(\phi_1) = -\frac{4b_1\ (\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)\ \phi_1^{\frac{w-2(\sigma+1)+k(\gamma+1)/\gamma}{k-\sigma-1}}}{\gamma\left[w-(\sigma+1-k/\gamma)\right]\left[w-2(\sigma+1)+k(\gamma+1)/\gamma\right](\gamma+1)^2}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} + \ldots ,$$

$$\rho_0(\phi_1) = \phi_1^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} - \frac{2(\sigma-k/\gamma)(\gamma-1)\ \ \phi_1^{-\left[\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}\right]}}{\gamma(\gamma+1)^2\alpha_0\left[2(\sigma+1)-k(\gamma+1)/\gamma\right]} + \ldots\right\},$$

$$\rho_0^{*}(\phi_1, r) = \phi_1^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} G_1(r) - \right.$$

$$\frac{2(\sigma-k/\gamma)(\gamma-1)\ G_3(r)}{\gamma\alpha_0\left[2(\sigma+1)-k(\gamma+1)/\gamma\right](\gamma+1)^2}\ \phi_1^{-\left[\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}\right]} -$$

$$\frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)\ J^{*}\ F_0(r)^{-1}}{\gamma\alpha_0(\gamma+1)^2(k-\sigma-1)^2\ r^{2(\sigma+1)-k(\gamma+1)/\gamma}}\ \phi_1^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}} +$$

$$\frac{\alpha_1(r)}{\gamma\alpha_0}\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} \phi_1^{\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\ \frac{F_0(r)^{-1+1/\gamma}}{r^{2(\sigma+1)-k(\gamma+1)/\gamma}} + \ldots\ldots ,$$

$$\rho_1(\phi_1) = \frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{\gamma\alpha_0(k-\sigma-1)^2\ (\gamma+1)^2}\ I^{*}\,\phi_1^{1+\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}} + \ldots\ldots ,$$

$$\rho_1^*(\phi_1,r) = \frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{\gamma\alpha_0(k-\sigma-1)^2\ (\gamma+1)^2}\ I^*\phi_1^{1+\frac{k(\gamma-1)}{\gamma(k-\sigma-1)}}\ G_3(r) + \ldots ,$$

$$\rho_2(\phi_1) = -\frac{2b_1}{\gamma}\ \phi_1^{\frac{w+k(\gamma-1)/\gamma}{k-\sigma-1}}\left\{\left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} +\right.$$

$$\frac{2(\sigma-k/\gamma)(\sigma+1-k/\gamma)(\gamma-1)}{\gamma\alpha_0(k-\sigma-1)^2\ (\gamma+1)^2}\left[\frac{(k-\sigma-1)^2}{\{w-(\sigma+1-k/\gamma)\}\{w-2(\sigma+1)+k(\gamma+1)/\gamma\}}\right.$$

$$\left.\left. - 1\right]\phi_1^{-\frac{2(\sigma+1)-k(\gamma+1)/\gamma}{k-\sigma-1}} + \ldots\ldots\right\} ,$$

for $k > (\sigma+1)$.

$$U_0(\phi_1) = 1 - \frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1 + \ldots\ldots ,$$

$$U_0^*(\phi_1,r) = -\frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1\ G_0(r) + \ldots\ldots ,$$

$$U_1(\phi_1) = -\frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^* + \ldots\ldots ,$$

$$U_1^*(\phi_1,r) = -\frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^*\,G_0(r) + \ldots\ldots ,$$

$$U_2(\phi_1) = -\frac{2b_1\,\sigma(\gamma-1)}{\gamma(\gamma+1)(\sigma+1-w)}\left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}}\phi_1^{1-\frac{w}{\sigma+1}} + \ldots\ldots ,$$

$$P_0(\phi_1) = \alpha_0 - \frac{\sigma(\gamma-1)}{(\gamma+1)^2(\sigma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^2 + \ldots\ldots ,$$

$$P_0^{*}(\phi_1, r) = \alpha_0 H_0(r) - \frac{\sigma(\gamma-1)}{(\gamma+1)^2(\sigma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} \phi_1^2\, G_0(r) + \ldots\ldots ,$$

$$P_1(\phi_1) = -\frac{2\sigma(\gamma-1)}{(\gamma+1)^2(\sigma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^{*} \phi_1 + \ldots\ldots ,$$

$$P_1^{*}(\phi_1, r) = -\frac{2\sigma(\gamma-1)}{(\gamma+1)^2(\sigma+1)} \left(\frac{b_0^2}{\alpha_0}\right)^{\frac{1}{\gamma}} I^{*} \phi_1\, G_0(r) + \ldots\ldots ,$$

$$P_2(\phi_1) = -\frac{4b_1\,\sigma(\gamma-1)(\sigma+1)}{\gamma(\gamma+1)^2(\sigma+1-w)\left[2(\sigma+1)-w\right]} \phi_1^{2-\frac{w}{\sigma+1}} + \ldots\ldots ,$$

$$\rho_0(\phi_1) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} - \frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)\alpha_0} \phi_1^2 + \ldots\ldots ,$$

$$\rho_0^{*}(\phi_1, r) = \left(\frac{\alpha_0}{b_0^2}\right)^{\frac{1}{\gamma}} G_1(r) - \frac{\sigma(\gamma-1)}{(\sigma+1)(\gamma+1)\alpha_0} \phi_1^2\, G_3(r) + \ldots\ldots ,$$

$$\rho_1(\phi_1) = -\frac{2\sigma(\gamma-1)}{\gamma(\sigma+1)(\gamma+1)\alpha_0} I^{*} \phi_1 + \ldots\ldots ,$$

$$\rho_1^{*}(\phi_1, r) = \frac{-2\sigma(\gamma-1)}{\gamma(\sigma+1)(\gamma+1)\alpha_0} I^{*} \phi_1\, G_3(r) + \ldots\ldots ,$$

$$\rho_2(\phi_1) = -\frac{4b_1\,\sigma(\gamma-1)(\sigma+1)}{\gamma(\gamma+1)(\sigma+1-w)\left[2(\sigma+1)-w\right]} \phi_1^{2-\frac{w}{\sigma+1}} + \ldots\ldots ,$$

for $k = 0$.

In the case $k = 0$ there may be some overlap between $U_0^*(\phi_1, r)$, $U_1(\phi_1)$ and $U_1^*(\phi_1, r)$ because of the lesser restrictions on the $f_i(r)$. The situation becomes more clear if this problem is treated as a case of $k < (\sigma+1)$.

In all cases we have used

$$G_0(r) = F_0(r)^{-1/\gamma} - 1 \quad ,$$

$$G_1(r) = F_0(r)^{1/\gamma} - 1 \quad ,$$

$$G_2(r) = \alpha_1(r) F_0(r)^{-1+1/\gamma} - \alpha_1(\infty) \quad ,$$

$$G_3(r) = F_0(r)^{-1} - 1 \quad ,$$

$$H_0(r) = F_0(r) - 1 \quad ,$$

$$H_1(r) = F_0(r)^{1-1/\gamma} - 1 \quad ,$$

$$H_2(r) = F_0(r)^{-1/\gamma} \left[r\frac{dF_0(r)}{dr} - kF_0(r) \right] + k \quad ,$$

$$H_3(r) = r\frac{dF_0(r)}{dr} - kF_0(r) + k \quad ,$$

$$H_4(r) = F_0(r)^{-1} \, r\frac{dF_0(r)}{dr} \quad .$$

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